

Four essays on the axiomatic method : cooperative game theory and scientometrics

Sylvain Ferrières

► To cite this version:

Sylvain Ferrières. Four essays on the axiomatic method : cooperative game theory and scientometrics. Economics and Finance. Université de Franche-Comté, 2016. English. NNT : 2016BESA0002 . tel-01450633

HAL Id: tel-01450633

<https://tel.archives-ouvertes.fr/tel-01450633>

Submitted on 31 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HHL



**ECOLE DOCTORALE
CARNOT - PASTEUR**

Dissertation

Submitted in fulfillment of the requirements for the degree

DOCTOR of ECONOMICS (*Dr. rer. oec.*)

at Besançon, on November 25th 2016

by

Sylvain Ferrières

Four essays on the axiomatic method: cooperative game theory and scientometrics

Assessor	P.U. Francis BLOCH	(Université Paris I Panthéon-Sorbonne, France)
Assessor	Prof. Hervé MOULIN	(University of Glasgow, Scotland)
Examinator	T.U. Encarnación ALGABA DURÁN	(Universidad de Sevilla, Spain)
Examinator	Prof. Dr. André CASAJUS	(HHL Leipzig, Germany)
Supervisor	Prof. Pierfrancesco LA MURA	(HHL Leipzig, Germany)
Supervisor	P.U. Sylvain BÉAL	(Univ. Bourgogne Franche-Comté, France)



Thèse effectuée au sein du **Centre de Recherche sur les Stratégies
Économiques**

de l'Université de Franche-Comté
30, avenue de l'Observatoire
BP 1559
25009 Besançon cedex
France

Cette thèse a fait l'objet d'un soutien financier de l'Université franco-allemande (UFA) quant aux dépenses liées à la cotutelle et du programme de recherche "DynaMITE: Dynamic Matching and Interactions: Theory and Experiments", contrat ANR-13-BSHS1-0010 quant aux déplacements en colloque.

Remerciements

En premier lieu, je remercie Sylvain Béal pour m'avoir initié à la théorie des jeux coopératifs et plus généralement à la recherche en économie, tout en me permettant de côtoyer cette communauté si ouverte des théoriciens des jeux. Il m'a soutenu dans les moments difficiles et guidé tout au long de cette thèse qui n'aurait pas vu le jour sans ses constants encouragements, sa rigueur et son énergie communicative.

Je souhaite exprimer ma gratitude à Philippe Solal et Éric Rémila qui, au travers de nos collaborations et lors de nombreux colloques, m'ont appris à canaliser mon tempérament dispersé et m'ont aidé à structurer mon travail. Mes plus sincères remerciements vont également à Frank Hüttner pour sa franche camaraderie et son aide indispensable dans la mise en place de la cotutelle, à tout le personnel administratif de l'École doctorale Carnot Pasteur et de la HHL, ainsi qu'à André Casajus et Pierfrancesco La Mura qui m'ont accueilli à Leipzig et beaucoup apporté scientifiquement.

Je remercie aussi Francis Bloch, Encarnación Algaba Durán et Hervé Moulin, pour l'intérêt qu'ils ont porté à mon travail et l'honneur qu'ils me font de participer au jury de ma thèse.

Je suis très reconnaissant envers mes collègues au labo ou à la fac, administratifs, enseignants, chercheurs, doctorants ou étudiants, qui m'ont permis de mener à bien ce long voyage en allégeant ma charge d'enseignement par leur bienveillance et leur compassion.

Je remercie enfin ma famille, mes filles Cyrielle et Mila, ainsi que mes proches pour leur présence, leur patience et leur réconfort. Merci particulièrement à Fanny pour son soutien indéfectible.

Contents

General introduction	1
Introduction générale	21
1 Axiomatic characterizations under players nullification	43
1.1 Introduction	45
1.2 Basic definitions and notations	48
1.2.1 Cooperative games with transferable utility	48
1.2.2 Values	49
1.2.3 Some axioms and existing characterizations	50
1.3 Player's nullification	51
1.4 Axiomatic study	54
1.4.1 Balanced contributions under nullification	56
1.4.2 Balanced collective contributions under nullification	59
1.4.3 Balanced cycle contributions under nullification	65
1.5 Revisiting the potential approach	71
1.6 Conclusion	73
2 Nullified equal loss property and equal division values	79
2.1 Introduction	81
2.2 Basic definitions and notations	83
2.2.1 Cooperative games with transferable utility	83
2.2.2 Values	84

2.2.3	Punctual and relational Axioms	85
2.3	Axiomatic study	86
2.3.1	General formula for efficient values satisfying the Nullified equal loss property	86
2.3.2	Linear symmetric and efficient values satisfying the Nullified equal loss property	89
2.3.3	Characterization of the class of convex combinations of ED and ESD	92
2.3.4	Punctual characterization of equal division values	93
2.4	Applications	94
2.4.1	Bargaining under risk	94
2.4.2	Softening the tragedy of the Commons	98
2.5	Concluding remarks	102
3	The proportional Shapley value and an application	107
3.1	Introduction	109
3.2	Definitions, notation and motivation	112
3.2.1	Notation	112
3.2.2	Cooperative games with transferable utility	113
3.2.3	Values	114
3.2.4	A motivating example: Land production economies	114
3.3	Legacy results	116
3.4	Main results	119
3.4.1	Potential, linearity and consistency	119
3.4.2	Proportional Balanced contributions under dummification . . .	129
3.5	Conclusion	131
4	An axiomatization of the iterated h-index and applications to sport rankings	149
4.1	Introduction	151

4.2	Index and iterated index	153
4.2.1	A richer class of indices	153
4.2.2	Operations on X	155
4.3	Axiomatic study	157
4.3.1	Axioms	157
4.3.2	Results	159
4.3.3	Discussion	162
4.4	Alternative sport rankings	163
4.4.1	Tennis	163
4.4.2	Basketball	166
4.4.3	Football	169
4.4.4	Discussion	170
4.5	Conclusion	171
General conclusion and future work		177
Appendix: Declarations of authorship		179
General bibliography		187

General introduction

*"Le plus grand plaisir humain est sans doute dans un travail difficile
et libre fait en coopération, comme les jeux le font assez voir."*

Alain (1928, XLIX, p. 108)

The axiomatic method in social sciences

Our human societies were built up from a back and forth tide of complex co-operations and conflicts throughout history. The allocation of scarce resources for instance is a frequent source of discord and often call on external arbitration. Simultaneously, such an external viewpoint is often needed when, facing a work requiring organization, a community has to relevantly proceed a division of labor. In order to overcome this kind of conflict of interest, many general principles have been debated and established, instituting piece by piece what became codes of law. A common trait of many such rules is the will to exceed their original issues, encompass counterfactual situations and reach a more universal range of applications. The objective is to rely on the principles themselves and not merely by defining a direct but ad hoc patch-up. Clearly formulated, they are submitted in thought to an impartial and rational spectator in an original position "behind a veil of ignorance" ([Rawls, 2009](#)). The resulting solutions will be strengthened and thus wholly accepted on the behalf of the chosen principles. This comprehensive and transparent approach, combined with a proper formalization, can be lumped into the axiomatic method. Born with euclidean geometry, it has spread to many fields of science, including nowadays social sciences. It requires to translate normative (or *a priori*) principles into formalized claims, called axioms. This process then allows to apprehend limits of these claims, derive general properties through logical deduction and test for

model internal consistency. Quoting [Thomson \(2001, p. 349\)](#):

"The objective of the axiomatic program is to give as detailed as possible a description of the implications of properties of interest, singly or in combinations, and in particular to trace out the boundary that separates combinations of properties that are compatible from combinations of properties that are not."

Typically, ethical concepts like fairness or efficiency are translated into mathematical requirements, trying to actually implement these welfarist or liberal principles. Moreover, when considering a particular framework, their translation into formalized requirements sensibly depends on the specifications of the class of problems at work. If the chosen solution relies on an extended class of problems, i.e. a set of situations close to and including the pending issue, everyone should agree on the need of these counterfactual circumstances. Though the process does not perform without concern and some requirements may be incompatible. But reasonable compromise should incorporate formal principles of distributive justice. This ideal was already described by Aristotle in *Eth. Nic.* , V. iii (6) (1131a, 20–27), transl. by [Rackham \(1934\)](#):

"And there will be the same equality between the shares as between the persons, since the ratio between the shares will be equal to the ratio between the persons; for if the persons are not equal, they will not have equal shares; it is when equals possess or are allotted unequal shares, or persons not equal equal shares, that quarrels and complaints arise."

Aristotle advocates that equals should be treated equally, and unequals unequally, in proportion to the relevant similarities and differences. Here the referee (think of a family court judge in a divorce settlement!) has to embody the word "relevant" with quantifiable tools: compensations (ex post equality), exogenous rights (ex ante inequalities), rewards (based on merit or unworthiness) and utilitarian adequacy (or fitness) of the share, for instance (see [Cook and Hegtvedt, 1983](#); [Deutsch, 1975](#), for similar taxonomies). Of course, these concepts are yet to be consistently defined within the context. In this thesis, we will circumscribe our domain of study to axiomatization of evaluative solutions to particular classes of multi-agent interaction problems to be defined below.

In economic and social sciences, the axiomatic approach has proved fruitful and many models have benefited from this formalization. The modeller, facing such social issues, tries to design a collection of logically independent properties, considered as desirable, that a solution to the problem should meet and, if possible, characterizes the set of solutions satisfying these properties. This method is inherently cross-disciplinary. Here are some examples of well-known applications.

- Social choice theory: this domain has given birth to one of the earliest achievements in the axiomatic approach to social science with the famous Arrow's impossibility theorem ([Arrow, 1963](#); [Kelly, 1978](#)). Its general purpose is to aggregate individual preferences or welfare into a collective choice, with concerns about representativeness –for instance in voting procedure (see [Balinski and Young, 1982](#), for an application in European Parliament), fairness –e.g. when choosing a level of a public good (see [Barbera and Jackson, 1994](#))– or impartiality –e.g. in nomination for a prize (see [Holzman and Moulin, 2013](#)).
- Conflicting claims: suppose that a group of agents have individual claims on a resource, but there is not enough of it to honor all demands. Bankruptcy problems (where the cash remainder of a bankrupted firm has to be shared between its creditors) and taxation problems (where a community decide how much each of its member should contribute in a public project) belong to such a situation (see [Thomson, 2003, 2015](#), for a comprehensive survey).
- Fair division, resource allocation and cost sharing: in many economic contexts the question of how to divide unproduced goods, costs or surplus generated by the cooperation of the agents is crucial (see [Aadland and Kolpin, 1998](#); [Ambec and Ehlers, 2008](#); [Ambec and Sprumont, 2002](#); [Moulin and Shenker, 1992](#); [Thomson, 2010](#), for instance). One main difficulty here resides in whether efficiency can be reconciled with equity.
- Bargaining problems: another great historical accomplishment in using the axiomatic approach is Nash's paper introducing the Nash's bargaining solution for two-person bargaining problems ([Nash, 1950](#)). A bargaining procedure is often understood as sequences of offers and counteroffers which specification has a great impact on the agreement that will come out of. Most of the time, in axiomatic studies, these procedures are reduced to some set of payoff vectors

so that players are only concerned with the selection of such a vector from this feasible set, according to their individual utility, the threat of ending up on a disagreement point and a combination of agreed rules.

- Indicators in various contexts: we only mention here some examples to highlight the field diversity in which indicator characterization was applied with success: centrality in network as to appraise the importance of a node in a graph (Bloch et al., 2016), scientometrics or influence measurement in order to assess the impact of scientific publications (Palacios-Huerta and Volij, 2004), riskiness which aims at objectively measuring the hazard of a gamble (Aumann and Serrano, 2008), poverty indicator (Foster, 2006; Sen, 1976), &c. Keep in mind that these indicators serve as actual decision tools in actual issues.
- Game theory: defined by Myerson (1991, p. 1) as the *"study of mathematical models of conflict and cooperation between intelligent rational decision-makers"*, game theory tries to gain some insight concerning human interactions and behaviors –the real-life *Homo sapiens*, by studying those of his mythical cousin, the *Homo rationalis*. An enlightening introduction is given by Aumann (1989). As a descriptive and predictive tool, it brings some operationally useful concepts such as Nash equilibrium (Nash, 1951) and its variants (Aumann, 1959, 1974; Harsanyi, 1968; Kreps and Wilson, 1982; Selten, 1965), for instance. These concepts, at the same time, may be considered as prescriptive and normative, giving advices for decision-makers; they have also been axiomatically characterized by general principles (Bernheim, 1998; Peleg and Tijs, 1996). As we will go into this subject in detail, we only mention here another essential milestone in the axiomatic approach history: the characterization by Shapley of his value for cooperative games with transferable utilities (Shapley, 1953).

Cooperative Game Theory

In this thesis, we will mainly focus on cooperative games, even if a chapter will apply the axiomatic method to another framework. These models of multi-agent interactions capture situations in which a set of players can make fully binding commitments (as agreements, promises or threats) as to form coalitions in order to

achieve a joint project. They do not explicit the procedure with which a coalition reaches its worth, neither answer the question how each player can best act strategically for her own benefit.

Instead, emphasis is put on two main questions: What coalition will form? How will they divide the available payoff between their members? Indeed, each coalition is not only pooling resources but may create an added value or an added cost, a synergy or a cacophony. Within this context, the modeller has to take into account the diversity of structures the cooperation could adopt and at the same time, like a referee, he has to provide simple intelligible rules for a fair sharing agreement. In this thesis, we leave out the strategic aspect of coalition formation and we also assume that players assess each coalition structure with the same value. This allows to model this general framework in a standard cooperative game with transferable utilities (TU-games henceforth) which summarizes the context by assigning a real number to every subset of players. This map is called a characteristic function and is supposed to contain all relevant information up to this point. A solution here is called an allocation rule (or value), and is efficient if it splits the complete worth of the grand coalition into an individual payoff function according to the characteristic function. The precise relationship between payoffs and characteristic functions is then the objective of the characterization process through axioms. Besides, it is not the utility as such that is transferred but commodities to which players indirectly attach the same utility. Money can serve this purpose as an infinitely divisible and desirable good. Generally, it is linearly related to utility so that, after a possible rescaling of utility functions, any transfer of it among players results in individual variations of utility which sum to zero (see [Aumann, 1960](#)).

Usually, an axiom imposes some specific changes or some invariance principle on the payoffs according to particular modifications of the game. These modifications may be thought from the original issue as counterfactual games: what if every worth is doubled? What if two players have exactly the same contributions? What if a player leaves the game, empty-handed or not, generating a reduced game? &c. so that the axiomatic study crucially relies on the class of such allowed counterfactual games to which axioms may apply.

Let us illustrate what we have presented so far by a simple but illuminating

class of cost allocation problems called airport games (see [Thomson, mimeo 2006; revised July 2013](#), for a comprehensive survey). The context is the following: suppose that, in an airport, a group of airlines companies are jointly using an airstrip. Having different fleets, their needs for the airstrip's length differ: the larger the planes, the longer the airstrip should be. If the airstrip accommodates a given plane, it also accommodates any smaller airplane. Suppose that the airstrip is large enough to accommodate the largest plane any airline flies, how should its cost be divided among the companies?

Denote by N the set of involved companies so that any $i \in N$ is characterized by the cost $c_i \in \mathbb{R}_+$ of the airstrip it needs. The needed cost is thus $\max_{i \in N} c_i$ and \mathcal{C}_N will denote the class of all such problems. Facing such a problem, an allocation rule has to be designed so as to satisfy desirable properties relying on the whole class \mathcal{C}_N , which therefore consists of all the counterfactual problems to be considered. A cost allocation $x = (x_i)_{i \in N} \in \mathbb{R}^N$ is defined so that $\sum_{i \in N} x_i = \max_{i \in N} c_i$ and $0 \leq x_i \leq c_i$ for all $i \in N$. An allocation rule φ maps to each problem $(N, c) \in \mathcal{C}_N$ a cost allocation $\varphi(N, c) \in \mathbb{R}^N$.

One great advantage of such a formalized model is that other economic contexts may belong to it with few changes if any and, once a solution is found for this class of problems, more complex variants and extensions may then be within reach. One natural allocation rule, used in the real world, is the so called **sequential equal contributions rule** SEC (discussed in [Baker and Associates, 1965](#)) which apply equal division between all players using a given segment and defined as following. Without loss of generality, we may suppose that $c_1 \leq c_2 \leq \dots \leq c_n$ and consider for each $i \in N$:

$$\text{SEC}_i(N, c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \dots + \frac{c_i - c_{i-1}}{n-i+1}. \quad (1)$$

This rule is indeed a cost allocation: the group $\{1, \dots, i\}$ contributes $i \cdot c_1/n + (i-1) \cdot (c_2 - c_1)/(n-1) + \dots + 1 \cdot (c_i - c_{i-1})/(n-i+1) \leq c_1 + (c_2 - c_1) + \dots + (c_i - c_{i-1}) = c_i$ so that $0 \leq \text{SEC}_i(N, c) \leq c_i$. Moreover the grand coalition N is paying exactly c_n . This allocation rule satisfies the following axioms:

- **Equal treatment of equals:** this axiom restricts the allocation rules when applied to a particular (counterfactual) situation. This kind of axiom is called

a punctual axiom. It imposes that if two players have equal needs, they should be paying the same amount. Formally, for each $c \in \mathcal{C}_N$ and each $i, j \in N$, if $c_i = c_j$ then $\varphi_i(N, c) = \varphi_j(N, c)$.

- **Independence of at-least-as-large costs:** this axiom expresses that a player's cost allocation should not depend on player's need of larger cost. Equivalently, her cost allocation should be the same in two problems only differing in the need of players having larger costs than her needs. As this axiom compares two different (counterfactual) situations, it is called a relational axiom. Formally, for each $i \in N$ and each $c, c' \in \mathcal{C}_N$ such that (a) $c_i = c'_i$, (b) the sets $\{j \in N, c_j < c_i\}$ and $\{j \in N, c'_j < c'_i\}$ are equal and (c) $c_j = c'_j$ whenever $c_j < c_i$, we have $\varphi_i(N, c) = \varphi_i(N, c')$.

Moreover, the last two axioms are in fact **characterizing** the sequential equal contributions rule within the class of cost allocation rules on \mathcal{C}_N : requiring these two axioms leads to this allocation rule solely ([Moulin and Shenker, 1992](#)). We have presented here a punctual and a relational axiom which are the main two types of axiom (see [Thomson, 2012](#)).

Interestingly, this class of problems may be embedded in the TU-game framework. For any $(N, c) \in \mathcal{C}_N$, considering any coalition $S \subseteq N$, we define the characteristic function $v_c(S) = \max_{i \in S} c_i$ so that (N, v_c) is a particular TU-game, modelling the problem equivalently. This process allows to consider allocation rules defined on TU-games. Mind that not all TU-games are generated by such a translation so that the preceding characterization is no more effective on the whole class of TU-games. Reciprocally, a characterization of an allocation rule on all TU-games may not apply in the smaller class of airport games (see [Thomson, 2001](#), for a general discussion on this topic). It happens however that the sequential equal distribution rule matches with one of the most famous allocation rule on TU-games (see [Littlechild and Owen, 1973](#)): the **Shapley value** Sh , which assigns to every player a payoff that measures, depending on the context, her productivity or her incentive to play the game, or even her aggregate bargaining power within such a TU-game.

In the sixty years elapsed since its first characterization, the Shapley value has been characterized in many very different ways (see [Myerson, 1980](#); [Young, 1985](#),

for instance) and has given birth to many extensions and particularizations –such as voting games (Dubey, 1975; Shapley and Shubik, 1954), games with a restricted set of coalitions (by a priori unions, in Owen (1977), or communication links, in Myerson (1977a) for instance), stochastic TU-games (Suijs and Borm, 1999), or for modelling coalitional externalities, TU-games in partition function form (Myerson, 1977b; Thrall and Lucas, 1963). It has been applied in various economic contexts: compensations in an auction of an indivisible good (Graham et al., 1990), fair division of unproduced goods with money (Moulin, 1992), rewards in multilevel marketing (Emek et al., 2011; Rahwan et al., 2014), queueing problems (Maniquet, 2003), logistics cost sharing (Lozano et al., 2013) and coalition formation (Laruelle and Valenciano, 2008) among others, but also in quite diverse fields (as depicted in Moretti and Patrone, 2008, and the subsequent comments) such as epidemiology (Gefeller et al., 1998), genetics (Moretti et al., 2007), reliability theory (Ramamurthy, 1990), pattern recognition (Grabisch, 1996) and statistics (Israeli, 2007).

Another sideways approach, complementing the axiomatic approach, is called the "Nash program". It consists in defining an explicit non-cooperative bargaining model yielding the given allocation rule as its outcome. The Shapley value has been analyzed from this point of view too (see Gul, 1989; Hart and Mas-Colell, 1996; Pérez-Castrillo and Wettstein, 2001). Perhaps some good reasons that makes this allocation rule so appealing –and often used as a normative tool– relies on the tremendous number of appealing properties it satisfies. To name but a few (taken from Roth, 1988), it is fundamentally based on a marginalistic principle (i.e. individual payoffs only depend on individual contributions to coalitions) and on a balanced contribution principle: a player's threat of leaving the game on another player's payoff is equal to the reversed threat by exchanging the role of the two chosen players.

Before dealing with a classical axiomatic characterization of the Shapley value, let us first give a formula on the whole class of TU-games \mathbb{V} . The formula (2), given earlier for airport games only, may be reformulated as following: instead of only considering the players' ordering according to their increasing needs c_i , consider any ordering $\sigma \in \mathcal{S}_N$ of the players, where \mathcal{S}_N denote the set of all orderings on the set N . According to this ordering σ , player $i \in N$ has to contribute

$m_i^\sigma(v) = v(\{j \in N, \sigma(j) < \sigma(i)\} \cup i) - v(\{j \in N, \sigma(j) < \sigma(i)\})$ ¹. In airport games, $m_i^\sigma(v) = \max_{j \text{ s.t. } \sigma(j) \leq \sigma(i)} c_j - \max_{j \text{ s.t. } \sigma(j) < \sigma(i)} c_j$ which may vary from 0 to c_i , depending on the coalition of preceding players.

Intuitively, players gather one by one in a room to form the grand coalition N , and each one who enters pays her marginal contribution if any. Then one may show that the sequential contribution rule SEC is equal to the average marginal contribution when all the different entering orders are equiprobable: $\mathbb{E}(m_i^\sigma(v))$. Called the "room parable", this allows to define the Shapley value on \mathbb{V} in the same way:

$$\text{Sh}_i(N, v) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_N} m_i^\sigma(v).$$

Now, denote by \mathbb{V}_N the set of all TU-games with a fixed set N of players. Let us consider one close characterization to Shapley's original one which involves four axioms (Shubik, 1962):

- **Efficiency:** this axiom imposes that the worth of the grand coalition has to be shared. For all $(N, v) \in \mathbb{V}$, one has $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- **Equal treatment of equals** (or Symmetry): this aforementioned axiom naturally extends from airport games to \mathbb{V} . For all $(N, v) \in \mathbb{V}$, all $i, j \in N$ such that $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, one has $\varphi_i(N, v) = \varphi_j(N, v)$.
- **Linearity:** For any finite player set N , the map $\varphi : \mathbb{V}_N \longrightarrow \mathbb{R}^N$ is linear.
- **Null player property:** this axiom explicits the treatment of a special kind of players, the null players. A null player in (N, v) is such that $v(S \cup i) = v(S)$ for any $S \subseteq N \setminus i$, i.e. she does not bring any extra worth (or cost) to any coalition she enters. This axiom requires that for all $(N, v) \in \mathbb{V}$, if i is a null player in (N, v) , then $\varphi_i(N, v) = 0$.

Stated equivalently, within the natural set of linear symmetric and efficient allocation rules, the Shapley value is the only allocation rule which satisfies the null player property, underlining the importance of the null players' treatment in this characterization. This last remark explains why many variants of the null player

¹Henceforth, a singleton $\{i\}$ is denoted i for simplicity.

property have been introduced in the literature to investigate new class of solutions, find alternative characterizations of the Shapley value and compare the Shapley value with other allocation rules through the change of as few axioms as possible. As we will deal with a closely related concept in the following pages, let us present some of them:

- **Dummy player property:** Instead of dealing with null players, this axioms deals with dummy players. A dummy player in (N, v) is such that $v(S \cup i) = v(S) + v(i)$ for any $S \subseteq N \setminus i$, i.e. she only adds her own worth to the coalition she is entering, no extra gain nor loss results from her cooperation. The axiom requires that for all $(N, v) \in \mathbb{V}$, if i is a dummy player in (N, v) , then $\varphi_i(N, v) = v(i)$.

Replacing the null player property by the dummy player property in the preceding characterization does not change the result.

- **Nullifying player property:** A nullifying player in (N, v) is such that $v(S) = 0$ for any coalition $S \ni i$. The axiom then requires that for all $(N, v) \in \mathbb{V}$, $\varphi_i(N, v) = 0$ if i is a nullifying player in (N, v) .

Replacing the null player property by the nullifying player property characterizes the **equal division value**² ED ([van den Brink, 2007](#)).

- **Null player in a productive environment property:** A productive environment arises when the grand coalition's worth is non negative. In this context, we may require that an allocation rule meets some solidarity principle and let null players get some non negative payoffs: for all $(N, v) \in \mathbb{V}$ such that $v(N) \geq 0$, $\varphi_i(N, v) \geq 0$ if i is a null player in (N, v) .

Using this axiom instead of the stronger null player property extends the characterization to a class of allocation rules containing the Shapley value and the equal division value, namely the egalitarian Shapley values (introduced by [Joosten, 1996](#)) Sh^α for $\alpha \leq 1$, where $\text{Sh}^\alpha = \alpha \text{Sh} + (1 - \alpha) \text{ED}$ (this result is due to [Casajus and Huettnner, 2013](#)).

²This allocation rule is defined by:

$$\text{ED}_i(N, v) = \frac{v(N)}{n}.$$

- **Null player out property:** For all $(N, v) \in \mathbb{V}$, if i is a null player in (N, v) , then for any $j \in N \setminus i$, $\varphi_j(N, v) = \varphi_j(N \setminus i, v|_{N \setminus i})$ where $v|_{N \setminus i}$ stands for the restriction of v to coalitions in $N \setminus i$.

This axiom states that removing a null player from the game does not change the payoffs of the other players (Derks and Haller, 1999). One can easily see that the combination of this axiom and efficiency implies the null player property.

- **Weak null player out property:** For all $(N, v) \in \mathbb{V}$, if i is a null player in (N, v) , then for any $j, k \in N \setminus i$, $\varphi_j(N, v) - \varphi_j(N \setminus i, v|_{N \setminus i}) = \varphi_k(N, v) - \varphi_k(N \setminus i, v|_{N \setminus i})$.

This axiom, introduced by van den Brink and Funaki (2009), is a weaker version of the preceding in the sense that it does not specify the payoff variation when a null player is removed –which is zero for the null player out property– but instead requires that the change in payoff should be the same for all remaining players. Both the Shapley value and the equal division value satisfy this axiom.

Here we are entering the heart of axiomatic studies: considering any characterization of an allocation rule of interest, what are the consequences of modifying –even slightly– one of the axioms under consideration? As we have seen, the treatment of null players is central in Shapley’s original axiomatization: adding no marginal contribution to any coalition, a null player gets a null payoff by the null player property. A substantial contribution of this thesis relies on other variants of the null player property allowing to include different principles of distributive justice and leading to different allocation rules, like the equal division rule for instance.

Summary of the thesis

This thesis is divided in four chapters which may be read independently. These chapters are directly drawn from working papers and each has its own notation. As a consequence, we are aware that some parts of this summary may seem redundant with the chapters’ introduction to the reader. This is the reason why a special effort is made in this section to discuss the links between the four following chapters. The

first two chapters of this thesis deal with the nullification operation defined below. The third chapter studies a non-linear variant of the weighted Shapley values, called proportional Shapley value. Finally, the fourth chapter is an illustration of another fruitful framework, dealing with evaluation of individual performance and ranking, in which the axiomatic approach is successfully exploited. Concerning the related literature on the different matters dealt with in this thesis, we rely on the articles' introduction themselves. We now present the main ideas and tools tackled in these works. Only briefly some results are reported here.

In the first chapter, published in *Mathematical Social Sciences* (Béal et al., 2016), we use the nullification operation in order to modify several popular relational axioms: balanced contributions (Myerson, 1980), balanced cycle contributions (Kamijo and Kongo, 2010) and balanced collective contributions (Béal et al., 2016). The common characteristic of these axioms is that they evaluate the consequences of removing a player from a TU-game on the payoff of some other players. For instance, balanced contributions requires, for any two players, equal allocation variation after the leave of the other player. As such the axiomatic study in Myerson (1980) operates on a class of TU-games with variable player sets. Together with the standard efficiency axiom, Myerson (1980) characterizes the Shapley value.

In our alternative approach, instead of leaving the game, a player stays in the game as a null player. More precisely, given $(N, v) \in \mathbb{V}_N$ and $i \in N$, we define $(N, v^i) \in \mathbb{V}_N$ so that $v^i(S) = v(S \setminus i)$ for any $S \subseteq N$. Player i becomes a null player in the new TU-game (N, v^i) : the worth of any coalition $S \ni i$ is now identical to that of the same coalition without her and the worth of the coalitions not containing her are left unchanged. This player is said to be nullified from (N, v) . It holds that $(v^i)^i = v^i$ and $(v^i)^j = (v^j)^i$ for $j \in N$. Thus, for any coalition $S \subseteq N$, the TU-game, denoted by (N, v^S) , obtained from (N, v) by the successive nullification of each player in S (in any order) is well-defined. There are various economic contexts in which the nullification of a player arises naturally. Let us develop one here (see Béal et al., 2016, for more examples).

Consider that a set N of people agree to carpool on a particular trip for some m specific days (see Naor, 2005). For each day $k = 1, \dots, m$, denote by D_k the subset

of people who show up this day and define the associated TU-game (N, v) so that the worth $v(S)$ of a coalition $S \subseteq N$ is equal to the total number of days such that at least one member of S showed up:

$$v(S) = |\{k = 1, \dots, m : D_k \cap S \neq \emptyset\}|.$$

Then suppose that, for any reason, participant i cannot eventually showed up during these days. The resulting associated TU-game (N, v') is thus modified so that $D'_k = D_k \setminus i$ for any day k . It holds clearly that $v' = v^i$, i.e. player i has been nullified from (N, v) .

The nullification operation has been introduced in [Neyman \(1989\)](#) to prove uniqueness of the Shapley value when applying the four aforementioned axioms characterizing the Shapley value ([Shubik, 1962](#)) to the additive group generated by a given TU-game (N, v) and the games resulting from all possible nullifications $(N, v^S)_{S \subseteq N}$. A different approach is considered in [Gómez-Rúa and Vidal-Puga \(2010\)](#) and [Béal et al. \(2014\)](#) where the authors measure the influence of nullifying a player in order to characterize the Shapley and the equal division values respectively. The major difference between erasing and nullifying a player is that the nullified player still stays in the TU-game. Therefore a natural question is the following: **is the impact of removing a player equivalent to keeping him nullified in the TU-game?** In this first chapter, we provide a systematic answer to this question by translating each of the aforementioned relational axioms, which we call "removal" axioms, into its corresponding "nullified" version. For instance, the nullified version of the axiom of balanced contributions requires, for any two players, equal allocation variation after the nullification of the other player. Below, we put emphasis on two interesting results in which this translation process brings different outcomes.

First, analogously to [Myerson \(1980\)](#), the combination of efficiency and balanced contributions under nullification characterizes not only the Shapley value but a family of allocation rules, corresponding to efficient affine Shapley values, i.e. equal to the Shapley value up to an exogenously added budget-balanced transfer scheme³ $a \in \mathbb{R}^N$. The second example relies on the characterization of the equal allocation of

³A budget-balanced transfer scheme $a \in \mathbb{R}^N$ is a vector such that $\sum_{i \in N} a_i = 0$.

non-separable costs⁴ EANSC by efficiency and the balanced collective contributions axiom (see [Béal et al., 2016](#)). This last axiom requires the same average impact of the removal of any player from a TU-game on the remaining players' payoff. Our similar result states that there exists a unique allocation rule that satisfies efficiency, equal treatment of equals (which is necessary here) and balanced collective contributions under nullification. This new allocation rule admits a closed form expression and, contrary to the equal allocation of non-separable costs, which possesses an egalitarian flavor, ends up to be extremely marginalistic: it endows the productive players even more than the Shapley value, at the expense of null players who compensate their payoffs.

The first chapter also examines the parallel to the potential approach ([Hart and Mas-Colell, 1989](#)) and proves that the original potential, based on the removal operation, is equal to its nullified version. Similarly to [Hart and Mas-Colell \(1989\)](#)'s characterization, we obtain that the Shapley value of a player in a TU-game is equal to the discrete derivative of the nullified potential of this TU-game with respect to this player's nullification. Thus, as in [Neyman \(1989\)](#), the potential approach is still operational on a given game and the games obtained from all its nullifications.

The second chapter of this thesis introduces a new axiom for TU-games with a fixed player set called the **nullified equal loss property**, dealing with player's nullification. The nullified equal loss property requires that if a player is nullified, then all other players experience the same payoff variation. Thus, this axiom possesses a solidarity flavor and this article plays around with it in two different ways.

In the first one, a comprehensive axiomatic study of this axiom, in combination with the efficiency axiom, suggests that it captures an essential feature of egalitarian allocation rules, such as the equal division and equal surplus division values (as opposed to marginalistic allocation rules studied in the first chapter). Indeed, a general formula is provided for the class of allocation rules satisfying both efficiency

⁴This allocation rule is defined by (see [Moulin, 1987](#), for instance):

$$\text{EANSC}_i(N, v) = v(N) - v(N \setminus i) + \frac{v(N) - \sum_{j \in N} (v(N) - v(N \setminus j))}{n}.$$

and the nullified equal loss property and a value φ within this class simplifies, up to an exogenously added budget-balanced transfer scheme between the players, to the equal division value when applied to any 0-normalized TU-game⁵. Let us state here, for once, this result more formally: an allocation rule φ on \mathbb{V} satisfies the nullified equal loss property and efficiency if and only if there are n functions $(F_i)_{i \in N}$ and n numbers $(a_i)_{i \in N}$ such that $\sum_{i \in N} a_i = 0$, $F_i(0) = 0$ for all $i \in N$ and:

$$\varphi_i(v) = a_i + F_i(v(i)) - \frac{1}{n-1} \sum_{j \in N \setminus i} F_j(v(j)) + \frac{v(N)}{n}.$$

As already mentioned, this leads to the simplified equation $\varphi_i(v^0) - a_i = \text{ED}_i(v^0)$ for any player $i \in N$. This general formula proves very useful to single out popular (class of) allocation rules. For instance, the class of linear combination of the equal surplus division and the equal division values corresponds to the class of linear symmetric and efficient allocation rules satisfying the nullified equal loss property. Moreover, the equal surplus division value⁶ is characterized by efficiency, the nullified equal loss property and the inessential game property. The later axiom requires that if all players are dummy players in a TU-game, i.e. that no synergy emerges from their cooperation, then all players are given their stand-alone worth.

The second part presents illustrations of the allocation rules involved so far in two very different economic contexts. Here, the nullified equal loss property is only invoked as a general principle but does not take part in a characterization process. The first one considers the nullification of a player as a random event in a context of bargaining under risk. It shows that the aforementioned class of egalitarian allocation rules allows to incorporate individual risk aversions in this context, thanks to a non-linear and heterogeneous particularization of the general formula above. The resulting solution of the risky bargaining problem leaves the players with the possibility to hedge their risky payoff by monetary transfers in order to reach the payoff of the certainty equivalent bargaining problem. The second application

⁵For any $(N, v) \in \mathbb{V}$, the 0-normalized TU-game v^0 is defined by $v^0(S) = v(S) - \sum_{i \in S} v(i)$ for any $S \subseteq N$.

⁶This allocation rule is defined by:

$$\text{ESD}_i(N, v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}.$$

aims at implementing by a strong Nash equilibrium the social welfare optimum in a non-cooperative model of a common-pool resource management. This equilibrium is reached without market or social planner, but thanks to a redistribution of the whole production, which corresponds to a specific convex combination of equal division value and equal surplus division value. This redistribution remunerates the players partly according to their individual efforts and partly by an equal pension levied through an internal tax. In a sense, this process softens the well-known tragedy of the Commons ([Hardin, 1968](#)) without involving a second good, like money, for facilitating transfers and without imposing a fixed production to each player.

The third chapter is devoted to the axiomatic study of a new allocation rule called the **proportional Shapley value** PSh. Our starting point is an alternative statement of the following Shapley value's formula:

$$\text{Sh}_i(N, v) = \sum_{S \ni i} \frac{\Delta_v(S)}{s}.$$

This formula involves the Harsanyi dividends $\Delta_v(S)$ ([Harsanyi, 1959](#)) which, roughly speaking, correspond to the "intrinsic worth" of a coalition S , after having recursively subtracted the intrinsic worth of all sub-coalitions of S :

$$\Delta_v(S) = v(S) - \sum_{T \subsetneq S} \Delta_v(T).$$

A well-known variant of the Shapley value (introduced in [Shapley, 1953](#)) is the weighted Shapley value, which splits the Harsanyi dividends in proportional to some exogenously positive weights $w = (w_i)_{i \in N}$ of its members instead of an equal split as in the Shapley value:

$$\text{Sh}_i^w(N, v) = \sum_{S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S).$$

The proportional Shapley value endogenizes these weights so that they are proportional to the players' stand-alone worth:

$$\text{PSh}_i(N, v) = \sum_{S \ni i} \frac{v(i)}{\sum_{j \in S} v(j)} \Delta_v(S).$$

The two last formulas are very similar so that all results stated on the class of weighted Shapley values which only involve fixed characteristic functions and theirs

subgames are still valid for PSh. For instance, as in [Myerson \(1980\)](#)'s characterization, PSh is characterized by efficiency and a proportional balanced contributions axiom. An adaptation of [Hart and Mas-Colell \(1989\)](#)'s potential function allows to characterize PSh by means of a proportional potential function. Finally, [Monderer et al. \(1992\)](#)'s result applies so that PSh belongs to the core of any convex TU-game in \mathcal{C}^0 .

Note that PSh is not defined on the whole class of TU-games \mathbb{V} . We restrict the domain to TU-games (N, v) such that every stand-alone worth $v(i)$ has the same sign; this class of TU-games is denoted by \mathcal{C}^0 . Note that there cannot exist any null player in \mathcal{C}^0 and, for a fixed player set N , the nullification operation takes us out of \mathcal{C}^0 . In order to characterize PSh in a comparable way to Sh, a dummification operation is defined in the same way: it corresponds to the loss of synergy of a player, becoming a dummy player. This new tool allows to translate removal axioms into dummified ones. In the same spirit as in the first two chapters, PSh is then characterized by efficiency, the dummified version of the proportional balanced contributions axiom and the inessential game property. Here, the inessential game property specifies the payoffs in the sub-class of additive TU-games and plays a comparable role as the null game axiom in the first chapter's characterization of the Shapley value. The two other axioms then extend the allocation rule in a unique way to the whole class \mathcal{C}^0 .

The class \mathcal{C}^0 may look restrictive but it actually includes many applications. Notably, we point out that PSh recommends a particularly relevant and natural payoff distribution in the context of an asymmetric land production economy, introduced by [Shapley and Shubik \(1967\)](#), in which the Shapley value is handled in a symmetric framework.

Contrary to generic weighted Shapley values, PSh satisfies the equal treatment of equals axiom but is not linear. Therefore, the classical characterization of the Shapley value has to be adapted accordingly in order to provide a similar characterization of PSh. As for the characterization with the dummification operation, the main results of this section rely on a similar approach. In a first step, we consider axioms satisfied by PSh so that their combination restricts the set of allocation rules

in the following way. Two allocation rules, which are equal on a small sub-class of \mathcal{C}^0 and satisfy the combination of axioms, are also equal on \mathcal{C}^0 . For instance, if two allocation rules are equal on quasi-additive TU-games⁷ and satisfy the combination of efficiency, dummy player out and a weak form of linearity then they are equal on \mathcal{C}^0 . The dummy player out axiom is an analogue to the null player out axiom discussed above. The weak linearity axiom only requires linearity within \mathcal{C}^0 for pairs of TU-games having proportional stand-alone worths. Note that both PSh and Sh satisfy these three axioms and while they are also equal on the class of additive TU-games in \mathcal{C}^0 , they differ on quasi-additive TU-games. A second step then specifies the allocation rule on this particular sub-class. For this purpose, we combine a consistency axiom and an axiom of standardness as in [Hart and Mas-Colell \(1989\)](#)'s characterization of the Shapley value. The consistency principle is fundamental in social sciences (see [Thomson, 2011](#), for a survey). A consistency axiom is a relational axiom which rests on the stability of an allocation rule when reducing a game. It usually imposes that when a group of players leaves the game with their payoffs, the remaining players' payoffs in the reduced game should be equal to those of the original game. A standardness axiom is a punctual axiom. It usually specifies the payoffs for two-player games which serve as a benchmark. In our case, we define a weaker consistency axiom, limited to quasi-additive games, and an axiom of proportional standardness so that PSh is characterized by the five aforementioned axioms. Moreover, replacing the proportional standardness by the classical standardness axiom yields a (comparable) characterization of the Shapley value on \mathcal{C}^0 .

The fourth and last chapter deals with a variant of the popular h -index ([Hirsch, 2005](#)), called the **iterated h -index** introduced in [García-Pérez \(2009\)](#). These indices attempt at quantifying socioeconomic phenomena such as reputation, influence, productivity or output's quality of scholars, journals or academic departments by relying on the citation number of scholar publications. More precisely, given a scholar endowed with n publications, her h -index is equal to the integer $h \leq n$ if h of her publications have at least h citations each, and her other publications have at most h citations each. The iterated h -index then consists of successive applications of the h -index on the remaining $n - h$ less quoted publications. The process

⁷A TU-game (N, v) is additive (resp. quasi-additive) if $v(S) = \sum_{i \in S} v(i)$ for all $S \subseteq N$ (resp. $S \subsetneq N$).

affords a multi-dimensional index which fixes one drawback of the h -index: it allows for lexicographic comparisons of scholars with the same h -index and gives a finer classification.

Citation analysis has quickly given birth to a great variety of ranking methods. Nowadays, as a widely used tool in human resources decisions, it has become crucial to actually justify their use from a scientific point of view. A growing literature seeks to relieve this breach by using the axiomatic approach on these ranking methods (see [Bouyssou and Marchant, 2014](#)). Our contribution in this chapter's first part lies in this movement: a characterization of the iterated h -index is provided and discussed in light of that of the h -index. Indeed, the iterated h -index is characterized by means of five axioms. Three axioms specify the behavior of multi-dimensional indices under natural operations: the first one states that, when the number of citations and the number of publications are multiplied by the same integer, the index is rescaled by this integer; the second one states that adding publications with a low number of citations does not change the first components of the index –which correspond to the most cited publications; the third one states that adding more citations to the most cited publications let the index unchanged. These axioms, already present in the literature for unidimensional indices, are extended here to multi-dimensional indices. As in the preceding chapter, a normalization axiom, which copes with the one citation case, and a consistency axiom, which states that reducing the set of publication to the lowest cited publications corresponds to depriving the index of its first components, allows to single out the iterated h -index. The consistency axiom is key to distinguish the iterated and the original h -index as, in our multi-dimensional framework, the h -index satisfies the four other axioms while violating this one.

The second part of the chapter gets back to the main function of such an index: inherently empirical, the iterated h -index is implemented on real data to assess performance in effective sports. Indeed, for two-players/teams games, replacing the list of publications by the list of matches won by a player or a team, and the number of citations of each publication by the number of match won by each player/team defeated by the studied player/team, allows to make use of the iterated h -index and define an alternative ranking. We present results in tennis, football and basketball and discuss whether the iterated h -index can be considered as a good proxy for

recovering ATP, french league and NBA rankings respectively.

Introduction générale

L'approche axiomatique en sciences sociales

Tout au long de l'Histoire, nos sociétés se sont construites dans une alternance chaotique entre une coopération complexe, fragile, et des affrontements parfois violents. Par exemple, l'allocation de ressources rares est une fréquente source de discordes qui nécessite souvent un arbitrage externe pour la résoudre. Parallèlement, lorsqu'il s'agit d'organiser la division du travail au sein d'une communauté souhaitant réaliser un projet d'envergure, un tel arbitrage est tout aussi utile. De nombreux principes généraux ont été débattus puis mis en oeuvre pour surmonter ces conflits d'intérêt, donnant petit à petit naissance à nos codes juridiques actuels. Ces règles de vie en société ont ceci en commun qu'elles prétendent dépasser les conflits initiaux pour inclure d'autres situations contrefactuelles et atteindre ainsi une portée plus générale. L'objectif est alors de faire reposer les décisions de l'arbitrage sur ces principes-mêmes, et non de trouver une solution directe mais *ad-hoc* ou éphémère. Ces préceptes sont soumis en pensée à un spectateur impartial et rationnel dans une position dite originelle, « derrière un voile d'ignorance » ([Rawls, 2009](#)), afin que les solutions en résultant soient acceptées au nom de ces principes, et ainsi plébiscitées. Cette approche, globale et transparente, combinée à une formalisation adéquate, forme ce que l'on peut rassembler sous la dénomination de « méthode axiomatique ». Née avec la géométrie euclidienne, elle s'est largement étendue à d'autres domaines scientifiques et, de nos jours, s'est développée avec succès en sciences sociales. Elle consiste à traduire chacun des principes normatifs en axiomes formalisés, ce qui permet d'appréhender leurs limites, de déduire des propriétés générales par des calculs logico-mathématiques, et de tester la cohérence des modèles faisant appel à eux. L'objectif du programme axiomatique est de donner une description aussi détaillée que possible des conséquences logiques de ces axiomes, seuls ou combinés.

En particulier, comme rappelle Thomson (2001, p. 349), il s'agit de tracer la frontière entre les axiomes qui, combinés, sont compatibles entre eux et ceux qui ne le sont pas. Typiquement, des concepts éthiques comme l'équité ou la quête de l'efficacité maximale sont transposés en propositions mathématiques afin d'être concrètement implémentés au sein de modèles théoriques. Cette traduction formelle dépend toutefois des spécificités de la classe de problèmes à résoudre et, si la solution choisie fait intervenir une classe plus générale – incluant celui que l'on cherche à dénouer, l'ensemble des participants doit convenir de la nécessité de faire intervenir ces problèmes contrefactuels. Ce procédé peut alors aboutir à des difficultés et il arrive parfois que certaines exigences soient incompatibles. Cependant un compromis raisonnable doit *a minima* mettre en application des principes de justice redistributive. Ce concept a déjà été souligné par Aristote dans l'*Éthique à Nicomaque*, V. iii (6) (1131a, 20–27), traduit par Tricot (1979) :

« Et ce sera la même égalité pour les personnes et pour les choses : car le rapport qui existe entre ces dernières, à savoir les choses à partager, est aussi celui qui existe entre les personnes. Si, en effet, les personnes ne sont pas égales, elles n'auront pas des parts égales ; mais les contestations et les plaintes naissent quand, étant égales, les personnes possèdent ou se voient attribuer des parts non égales, ou quand, les personnes n'étant pas égales, leurs parts sont égales. »

Aristote préconise le traitement égalitaire des égaux et inégalitaire des inégaux, en proportion des similarités et des différences pertinentes. L'arbitrage doit alors proposer un sens concret et quantifiable à ces différences en ayant par exemple recours à des compensations, pour rétablir l'équité *a posteriori*, à des droits exogènes, pour limiter *a priori* les iniquités, à des récompenses ou des punitions, fondées sur le mérite ou les torts, et à des procédés de partage prenant en compte l'utilité sociale. Évidemment, ces concepts doivent encore être définis de manière cohérente en fonction du contexte. Dans cette thèse, nous limitons notre étude à l'axiomatisation de solutions, dites ponctuelles, concernant certaines classes de problèmes d'interactions entre plusieurs agents, que nous définirons plus bas.

En économie et en sciences sociales, l'approche axiomatique s'est révélée fructueuse et de nombreux modèles ont bénéficié de cette formalisation. Face à un problème social, le modélisateur tente de concevoir une collection de propriétés,

logiquement indépendantes et considérées comme souhaitables, que devrait réunir une solution au problème et, si possible, de caractériser l'ensemble des solutions répondant à ces propriétés. Cette méthode est intrinsèquement interdisciplinaire. Voici quelques exemples d'applications bien connues :

- en théorie du choix social : ce domaine a donné naissance à l'une des premières réalisations de l'approche axiomatique en science sociale avec le fameux théorème d'impossibilité d'Arrow ([Arrow, 1963](#); [Kelly, 1978](#)). Son objectif général est d'agrèger les préférences ou le bien-être individuels dans un choix collectif, en tenant compte de la représentativité – par exemple dans la conception de procédure de vote (comme [Balinski and Young, 1982](#), dans le cas du Parlement européen), de l'équité – comme par exemple dans le choix d'un niveau de qualité ou de couverture d'un bien public (voir [Barbera and Jackson, 1994](#)), ou de l'impartialité – comme dans la nomination pour un prix (voir [Holzman and Moulin, 2013](#)).
- dans le cas de revendications rivales : supposons que chaque agent d'un groupe ait des prétentions sur une ressource, mais que celle-ci ne soit pas en quantité suffisante pour honorer toutes les demandes. Les problèmes de banqueroute (où le solde d'une entreprise en faillite doit être partagé entre ses créanciers) et les problèmes d'imposition (où une communauté doit décider combien chacun de ses membres devrait contribuer à un projet public) appartiennent à un tel cadre (voir [Thomson, 2003, 2015](#), pour une revue détaillée de la littérature).
- dans le principe d'équité concernant le partage des coûts ou des ressources : dans de nombreux contextes économiques, la méthode d'allocation des biens non produits, des coûts ou du surplus générés par la coopération des agents constitue une question cruciale (voir [Aadland and Kolpin, 1998](#); [Ambec and Ehlers, 2008](#); [Ambec and Sprumont, 2002](#); [Moulin and Shenker, 1992](#); [Thomson, 2010](#), par exemple). Une des difficultés principales réside dans le fait de concilier équité et parcimonie.
- en théorie de la négociation : l'article de Nash ([Nash, 1950](#)) introduisant la solution de Nash pour les problèmes de négociation à deux joueurs est un autre grand accomplissement de l'approche axiomatique. Une procédure de négociation est souvent considérée comme une séquence d'offres et de contre-offres dont la spécification a un impact sur l'accord qui en résultera. La plupart

du temps, dans l'étude axiomatique, ces procédures sont réduites à un ensemble de vecteurs de gains accessibles de sorte que les joueurs sont concernés uniquement par la sélection d'un tel vecteur dans cet ensemble. La solution dépendra de leur utilité individuelle, de la menace de se retrouver sur un point de désaccord, et d'une combinaison de principes convenue à l'avance.

- dans l'élaboration d'indicateurs dans divers contextes : nous ne mentionnerons ici que quelques exemples pour mettre en évidence la diversité des domaines dans lequel la caractérisation axiomatique de ces indicateurs a été fructueuse : la mesure de centralité dans un réseau pour évaluer l'importance d'un noeud dans un graphe (Bloch et al., 2016), la scientométrie ou la mesure d'influence afin d'évaluer l'impact des publications scientifiques (Palacios-Huerta and Volij, 2004), la mesure de risque qui vise à évaluer objectivement le risque d'un pari (Aumann and Serrano, 2008), les indicateurs de pauvreté (Foster, 2006; Sen, 1976), etc. Il faut garder à l'esprit que ces indicateurs servent d'outils d'aide à la décision dans des situations réelles.
- en théorie des jeux : définie par Myerson (1991) comme l'étude de modèles mathématiques de conflits et de coopération entre agents rationnels, la théorie des jeux tente d'appréhender les comportements humains – de l'*Homo sapiens* bien réel, à travers ceux de son cousin mythique, l'*Homo rationalis*. En tant qu'outil descriptif et prédictif, des concepts opérationnels, tels que l'équilibre de Nash (Nash, 1951) et ses nombreuses variantes (Aumann, 1959, 1974; Harsanyi, 1968; Kreps and Wilson, 1982; Selten, 1965), ont été développés et étudiés en profondeur. Ces concepts peuvent aussi être considérés comme prescriptifs et normatifs, et servir de point de départ dans un cadre d'aide à la décision ; ils ont également été axiomatisés (Bernheim, 1998; Peleg and Tijs, 1996). Comme nous allons entrer dans ce sujet en détail, nous ne citons ici qu'un autre jalon essentiel dans l'histoire de l'approche axiomatique : la caractérisation par Shapley de sa valeur pour les jeux coopératifs à utilité transférable (Shapley, 1953).

Théorie des jeux coopératifs

Cette thèse porte essentiellement sur l'étude axiomatique des jeux coopératifs, même si un chapitre appliquera la méthode axiomatique à un autre cadre. Les jeux coopératifs modélisent des situations d'interactions entre agents qui peuvent contractualiser des engagements. Ces contrats sont alors contraignants et les agents peuvent ainsi former des coalitions afin de parvenir à réaliser un projet commun. La description du procédé par lequel une coalition coordonne ses actions et la question de la meilleure stratégie individuelle dans ce cadre ne font pas partie du modèle. L'accent est mis sur deux questions principales : quelle coalition va se former ? Comment les membres de cette coalition vont-ils diviser les bénéfices ou les pertes issus de leur collaboration ? Chaque coalition met non seulement en commun ses ressources, mais peut en effet créer une valeur ajoutée (ou au contraire un coût supplémentaire), selon la synergie ou la désorganisation de ses membres. Dans ce contexte, le modélisateur doit prendre en compte la diversité des structures que cette coopération peut adopter et en même temps, comme un arbitre, il doit fournir des règles simples et intelligibles pour parvenir à un accord sur un partage équitable. Dans cette thèse, nous laissons de côté l'aspect stratégique de la formation des coalitions – et notamment de la grande coalition contenant tous les joueurs – et nous supposons également que les joueurs estiment pareillement la valeur créée par chaque coalition. Ces hypothèses définissent le cadre général des jeux coopératifs standards à utilité transférable (ou jeux TU) qui modélisent une telle situation en attribuant un nombre réel ou capacité à chaque sous-ensemble de joueurs. Cette application est appelée fonction caractéristique et contient toutes les informations pertinentes. Une solution est ici appelée règle de répartition (ou simplement valeur) et est dite efficiente si elle partage la capacité de la grande coalition en un vecteur individuel de gains en fonction de la fonction caractéristique. L'objectif de la caractérisation axiomatique consiste maintenant en la description de la relation précise entre gains et fonctions caractéristiques. Par ailleurs, ce n'est pas l'utilité en tant que telle qui est transférée, mais, par exemple, un bien ou une ressource auquel les joueurs attachent indirectement la même utilité. La monnaie peut servir à de tels transferts en tant que bien désirable et infiniment divisible car, en général, l'utilité dépend du capital monétaire de manière linéaire. Ainsi, tout transfert monétaire entre les joueurs peut être considéré comme un transfert d'utilité dont la somme des variations est nulle, quitte à modifier les fonctions d'utilité individuelles par une transformation affine.

Généralement, un axiome se présente sous la forme d'une modification des gains des joueurs ou, au contraire, d'une invariance lorsque le jeu subit une transformation spécifique. Ces transformations peuvent être considérées comme des opérations menant à des situations contrefactuelles à partir d'un problème donné : comment doivent évoluer les gains si la capacité de chaque coalition est doublée ? Que faire si deux joueurs ont exactement les mêmes contributions lorsqu'ils entrent dans une coalition ? Que faire si un joueur quitte le jeu, en partant avec ou sans une rétribution, générant ainsi un jeu réduit ? etc. L'étude axiomatique repose alors essentiellement sur la définition de la classe de jeux contrefactuels autorisés, à laquelle ces axiomes peuvent s'appliquer.

Illustrons maintenant ce que nous avons présenté par une classe simple, mais éclairante, de problèmes de répartition des coûts appelés jeux d'aéroport. Le contexte est le suivant : supposons que, dans un aéroport, un groupe de compagnies aériennes utilise conjointement une piste d'atterrissage. Ayant des flottes différentes, elles ont des besoins différents quant à la longueur de la piste d'atterrissage : plus les avions sont grands, plus la piste d'atterrissage devra être longue. Si la piste d'atterrissage est adaptée à un avion de taille donnée, elle peut aussi servir pour un avion plus petit. Supposons maintenant que la piste d'atterrissage soit assez grande pour accueillir le plus grand avion de toutes les compagnies aériennes réunies, comment son coût doit être réparti parmi ces compagnies ?

Désignons par N l'ensemble des compagnies impliquées et pour chaque compagnie $i \in N$, par $c_i \in \mathbb{R}_+$ le coût de la piste d'atterrissage dont elle seule a besoin. Le coût total à répartir est donc égal à $\max_{i \in N} c_i$. Nous noterons \mathcal{C}_N cette classe de problèmes de répartition de coût. Ainsi, pour résoudre un tel problème, nous devons déterminer une règle d'allocation vérifiant des propriétés souhaitables, à définir, mais ces propriétés devront s'appliquer à la classe \mathcal{C}_N toute entière, car celle-ci forme l'ensemble des problèmes contrefactuels à considérer. Une répartition des coûts $x = (x_i)_{i \in N} \in \mathbb{R}^N$ doit partager le coût total : $\sum_{i \in N} x_i = \max_{i \in N} c_i$ et chaque part doit aussi satisfaire la condition de participation : $0 \leq x_i \leq c_i$ pour tout $i \in N$. Une règle d'allocation φ est alors une fonction qui, à chaque problème $(N, c) \in \mathcal{C}_N$, associe une répartition des coûts $\varphi(N, c) \in \mathbb{R}^N$.

Nous nous intéressons maintenant à une règle d'allocation particulièrement

intuitive – et qui a été réellement utilisée dans ce cadre (voir [Baker and Associates, 1965](#)) : la règle de répartition séquentielle égalitaire SEC. Elle divise de manière égale le coût de chaque portion de la piste entre les joueurs qui en ont besoin. Pour la définir, nous pouvons supposer, sans perte de généralité, que les joueurs sont classés selon leur besoin, par ordre croissant de coût : $c_1 \leq c_2 \leq \dots \leq c_n$. Pour chaque $i \in N$, posons :

$$\text{SEC}_i(N, c) = \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \dots + \frac{c_i - c_{i-1}}{n-i+1}. \quad (2)$$

Cette règle définit bien une répartition des coûts car chaque groupe $\{1, \dots, i\}$ contribue à hauteur de $i \cdot c_1/n + (i-1) \cdot (c_2 - c_1)/(n-1) + \dots + 1 \cdot (c_i - c_{i-1})/(n-i+1) \leq c_1 + (c_2 - c_1) + \dots + (c_i - c_{i-1}) = c_i$ de telle sorte que $0 \leq \text{SEC}_i(N, c) \leq c_i$ et, de plus, la grande coalition N paye exactement $c_n = \max_{i \in N} c_i$. Cette règle d'allocation vérifie en outre les deux propriétés suivantes :

- **Traitement égalitaire des égaux** : cet axiome impose une restriction dans une situation (contrefactuelle) particulière. Ce type d'axiome est alors dit « ponctuel ». Plus précisément, il impose que, dans le cas où deux joueurs ont les mêmes besoins, ils doivent contribuer au projet de manière identique. Formellement, pour chaque problème $c \in \mathcal{C}_N$ et chaque paire de joueurs $i, j \in N$, si $c_i = c_j$ alors $\varphi_i(N, c) = \varphi_j(N, c)$.
- **Insensibilité à l'ampleur des plus grandes demandes** : cet axiome exprime que la part payée par un joueur ne doit pas dépendre du coût de construction d'une piste plus grande que celle dont il a besoin. De manière équivalente, sa contribution au projet doit être la même dans deux situations qui ne diffèrent que par les coûts de joueurs ayant des besoins supérieurs au sien. Ce type d'axiome compare deux situations (contrefactuelles) différentes et est alors dit « relationnel ». Formellement, pour chaque joueur $i \in N$ et deux problèmes $c, c' \in \mathcal{C}_N$ tels que (a) $c_i = c'_i$, (b) les ensembles $\{j \in N, c_j < c_i\}$ et $\{j \in N, c'_j < c'_i\}$ sont égaux et (c) $c_j = c'_j$ lorsque $c_j < c_i$, nous avons $\varphi_i(N, c) = \varphi_i(N, c')$.

Ces deux propriétés **caractérisent** en fait la règle de répartition séquentielle égalitaire dans la classe des règles d'allocation définies sur \mathcal{C}_N : si ces deux propriétés sont requises, le partage doit s'effectuer selon cette règle d'allocation ([Moulin and Shenker, 1992](#)). Nous avons présenté ici un axiome ponctuel et un axiome relation-

nel. La plupart des axiomes que nous rencontrerons peuvent se ranger dans une de ces deux catégories (voir [Thomson, 2012](#)).

Remarquons que les problèmes que nous venons de présenter peuvent être représentés par des jeux TU. En effet, étant donné $(N, c) \in \mathcal{C}_N$, nous définissons la capacité $v_c(S) = \max_{i \in S} c_i$ pour chaque coalition $S \subseteq N$, et (N, v_c) modélise le problème par un jeu TU de manière équivalente. Ce procédé permet de considérer les solutions données par des règles d'allocation définies sur les jeux TU. Toutefois, tous les jeux TU ne sont pas des jeux d'aéroport. Ainsi, la caractérisation précédente ne s'applique pas à l'ensemble des jeux TU. Réciproquement, une caractérisation d'une règle d'allocation sur la classe des jeux TU peut ne plus être valide sur la classe restreinte des jeux d'aéroports (voir [Thomson, 2001](#), pour une discussion sur ce sujet). Il se trouve cependant que la règle de répartition séquentielle égalitaire correspond à l'une des règles d'allocation les plus connues sur la classe des jeux TU (voir [Littlechild and Owen, 1973](#)) : la **valeur de Shapley** notée Sh , qui associe à chaque joueur un paiement qui reflète, selon le contexte, sa productivité ou son intérêt à participer au jeu, ou encore une agrégation de son pouvoir de négociation au sein de chaque coalition.

En soixante ans, depuis sa caractérisation initiale, la valeur de Shapley a été caractérisée de nombreuses fois par des propriétés très différentes (entre autres, [Myerson, 1980](#); [Young, 1985](#)) et a donné naissance à de multiples extensions et des cas particuliers comme les jeux de vote ([Dubey, 1975](#); [Shapley and Shubik, 1954](#)), les jeux comportant des restrictions sur la formation des coalitions (par exemple : les jeux de graphe, dans [Myerson \(1977a\)](#) ou avec unions *a priori*, dans [Owen \(1977\)](#)), jeux TU aléatoires ([Suijs and Borm, 1999](#)), ou, pour modéliser les externalités liées à la formation de coalitions adverses, les jeux TU sous forme de fonction de partitions ([Myerson, 1977b](#); [Thrall and Lucas, 1963](#)). Cette valeur a été appliquée dans de nombreux contextes économiques : dans le calcul des compensations lors d'une enchère d'un bien indivisible ([Graham et al., 1990](#)), pour un partage équitable de biens non-produits avec transferts monétaires ([Moulin, 1992](#)), dans le calcul des rétributions pour un marché développé par parrainage ([Emek et al., 2011](#); [Rahwan et al., 2014](#)), pour les problèmes de file d'attente ([Maniquet, 2003](#)), de partage de coûts logistiques ([Lozano et al., 2013](#)) et de formation des coalitions ([Laruelle and Valenciano, 2008](#)),

etc. La valeur de Shapley a aussi été utilisée dans bien d'autres domaines (listés dans [Moretti and Patrone, 2008](#), et les commentaires subséquents) : en épidémiologie ([Gefeller et al., 1998](#)), en génétique ([Moretti et al., 2007](#)), en fiabilité ([Ramamurthy, 1990](#)), en reconnaissance de formes ([Grabisch, 1996](#)) et en statistique ([Israeli, 2007](#)).

Une approche complémentaire à l'approche axiomatique, le « programme de Nash », s'est aussi intéressé à la valeur de Shapley (voir [Gul, 1989](#); [Hart and Mas-Colell, 1996](#); [Pérez-Castrillo and Wettstein, 2001](#)). Il consiste, pour une règle d'allocation définie sur les jeux TU, à définir explicitement un modèle de négociation non-coopératif dont les paiements des joueurs à l'équilibre correspondent à la règle étudiée. Le nombre impressionnant de propriétés vérifiées par la valeur de Shapley est probablement à l'origine de son utilisation fréquente comme référence normative. Citons-en deux fameuses (tirées de [Roth, 1988](#), consacré à la valeur de Shapley) : tout d'abord, c'est une valeur fondamentalement marginaliste (la part d'un joueur ne dépend que de sa contribution propre à chaque coalition qu'il rejoint) et qui vérifie le principe de contributions équilibrées : la variation de paiement d'un joueur i lorsqu'un autre joueur j quitte le jeu est égale à la variation de paiement du joueur j lorsque i quitte le jeu, la menace de quitter le jeu qu'un joueur peut faire subir à un autre joueur est équilibrée entre ces joueurs.

Avant d'exposer une caractérisation classique de la valeur de Shapley, nous allons donner une formule de cette valeur sur l'ensemble \mathbb{V} de tous les jeux TU. La formule (2), présentée plus haut dans le cadre des jeux d'aéroports, peut être reformulée de la manière suivante : au lieu de classer les joueurs par ordre croissant des besoins c_i et les faire payer successivement pour chaque tronçon de piste correspondant à leur besoin, nous pouvons considérer n'importe quel ordre $\sigma \in \mathcal{S}_N$, où \mathcal{S}_N désigne l'ensemble des ordres possibles sur N et faire construire (et payer) le tronçon de piste à chaque joueur en fonction de cet ordre d'arrivée. Ainsi, si un joueur a besoin d'une plus petite piste d'atterrissage qu'un autre joueur mais arrive après lui dans l'ordre σ , il n'aura rien à payer. Ainsi, sa contribution vaut maintenant : $m_i^\sigma(v) = v(\{j \in N, \sigma(j) < \sigma(i)\} \cup i) - v(\{j \in N, \sigma(j) < \sigma(i)\})$ ⁸. Dans les jeux d'aéroport, nous obtenons $m_i^\sigma(v) = \max_{j \text{ t.q. } \sigma(j) \leq \sigma(i)} c_j - \max_{j \text{ t.q. } \sigma(j) < \sigma(i)} c_j$ qui peut varier entre 0 et c_i , selon les joueurs précédant l'arrivée du joueur i .

⁸Un singleton $\{i\}$ sera désormais noté i .

Intuitivement, chaque joueur entre dans une salle pour former la grande coalition N en apportant sa contribution marginale, éventuellement nulle, à la construction de la piste. On peut alors montrer que la règle de répartition séquentielle égalitaire SEC est égale à la moyenne de ces contributions marginales lorsque tous les ordres σ sont pris en compte avec équiprobabilité : $\mathbb{E}(m_i^\sigma(v))$. Cela permet de définir la valeur de Shapley sur \mathbb{V} de manière identique :

$$\text{Sh}_i(N, v) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_N} m_i^\sigma(v).$$

Soit \mathbb{V}_N l'ensemble des jeux TU dans lesquels l'ensemble des joueurs est fixé et égal à N . La valeur de Shapley est caractérisée par les quatre axiomes suivants (cette caractérisation, due à [Shubik \(1962\)](#), est proche de la caractérisation originale par Shapley) :

- **Efficience** : cet axiome impose que la valeur répartisse exactement la capacité de la grande coalition. Pour tout $(N, v) \in \mathbb{V}$, on a $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- **Traitement égalitaire des égaux** (ou Symétrie) : cet axiome – déjà mentionné pour les jeux d'aéroports – s'étend naturellement à \mathbb{V} . Pour tout $(N, v) \in \mathbb{V}$, et toute paire $i, j \in N$ de joueurs tels que $v(S \cup i) = v(S \cup j)$ pour tout $S \subseteq N \setminus \{i, j\}$, on a $\varphi_i(N, v) = \varphi_j(N, v)$.
- **Linéarité** : Pour tout ensemble fini N de joueurs, l'application $\varphi : \mathbb{V}_N \longrightarrow \mathbb{R}^N$ est linéaire.
- **Propriété du joueur nul** : cet axiome précise le traitement d'un type important de joueurs : les joueurs nuls. Un joueur nul dans (N, v) est tel que $v(S \cup i) = v(S)$ pour tout $S \subseteq N \setminus i$, *i.e.* il n'apporte ni plus-value ni coût à une coalition qu'il rejoint. Cet axiome impose que pour tout $(N, v) \in \mathbb{V}$, et tout joueur nul i dans (N, v) , on ait $\varphi_i(N, v) = 0$.

Une autre manière d'énoncer cette caractérisation consiste à dire que la valeur de Shapley est l'unique règle d'allocation linéaire, symétrique et efficiente (ces trois propriétés étant assez naturelles) qui satisfait la propriété du joueur nul. Cette constatation souligne l'importance des joueurs nuls dans cette caractérisation et explique pourquoi de nombreuses variantes de cette propriété ont été introduites dans

la littérature afin d'établir des caractérisations alternatives de la valeur de Shapley, ou de caractériser de nouvelles classes de règles d'allocation et les comparer à la valeur de Shapley en gardant en commun le plus d'axiomes possible.

Comme les deux premiers chapitres de ce document exposent des concepts proches, nous présentons ici quelques variantes de la propriété du joueur nul :

- **Propriété du joueur neutre** : cette propriété précise le traitement d'un autre type de joueur. Un joueur neutre dans (N, v) est tel que $v(S \cup i) = v(S) + v(i)$ pour tout $S \subseteq N \setminus i$, *i.e.* le joueur ne fait qu'ajouter sa capacité propre à celle d'une coalition qu'il rejoint, il ne résulte ni plus-value ni perte de sa coopération. L'axiome impose que pour tout $(N, v) \in \mathbb{V}$, si i est un joueur neutre dans (N, v) , alors $\varphi_i(N, v) = v(i)$.

La caractérisation précédente reste inchangée si la propriété du joueur nul est remplacée par la propriété du joueur neutre.

- **Propriété du joueur nullifiant** : un joueur nullifiant dans (N, v) est tel que $v(S) = 0$ pour toute coalition $S \ni i$. Cet axiome impose que pour $(N, v) \in \mathbb{V}$, $\varphi_i(N, v) = 0$ si i est un joueur nullifiant dans (N, v) .

Le truchement de la propriété du joueur nul par la propriété du joueur nullifiant, permet de caractériser le **partage égalitaire**⁹ ED ([van den Brink, 2007](#)).

- **Propriété du joueur nul en environnement productif** : on définit un environnement productif lorsque la capacité de la grande coalition est positive ou nulle. Dans ce contexte, on peut rechercher à ce que la règle d'allocation satisfasse un principe de solidarité en laissant aux joueurs nuls une rétribution positive ou nulle : pour tout $(N, v) \in \mathbb{V}$ tel que $v(N) \geq 0$, $\varphi_i(N, v) \geq 0$ si i est un joueur nul dans (N, v) .

En utilisant cet axiome à la place de la propriété du joueur nul, qui est plus forte, on étend la caractérisation à une classe de règles d'allocation, appelées les valeurs de Shapley égalitaires $\text{Sh}^\alpha = \alpha \text{Sh} + (1 - \alpha) \text{ED}$ (introduites par [Joosten](#),

⁹Cette valeur est définie par :

$$\text{ED}_i(N, v) = \frac{v(N)}{n}.$$

1996) pour $\alpha \leq 1$, et contenant à la fois la valeur de Shapley et le partage égalitaire (ce résultat est dû à [Casajus and Huettner, 2013](#)). .

- **Propriété du retrait d'un joueur nul** : pour tout $(N, v) \in \mathbb{V}$, si i est un joueur nul dans (N, v) , alors pour tout $j \in N \setminus i$, $\varphi_j(N, v) = \varphi_j(N \setminus i, v|_{N \setminus i})$ où $v|_{N \setminus i}$ désigne la restriction de v aux coalitions incluses dans $N \setminus i$.

Cet axiome impose que le retrait d'un joueur nul du jeu ne change pas les rétributions des autres joueurs ([Derks and Haller, 1999](#)). Il est facile de voir que cet axiome, combiné à l'efficacité, implique la propriété du joueur nul.

- **Propriété affaiblie du retrait d'un joueur nul** : pour tout $(N, v) \in \mathbb{V}$, si i est un joueur nul dans (N, v) , alors pour tout $j, k \in N \setminus i$, $\varphi_j(N, v) - \varphi_j(N \setminus i, v|_{N \setminus i}) = \varphi_k(N, v) - \varphi_k(N \setminus i, v|_{N \setminus i})$.

Cet axiome, introduit par [van den Brink and Funaki \(2009\)](#), est une forme plus faible de l'axiome précédent car il ne précise pas quelle est la variation de paiement lorsqu'un joueur nul est retiré du jeu, sinon qu'elle doit être la même pour tous les joueurs restants, contrairement à la propriété du retrait du joueur nul qui la fixe égale à zéro. La valeur de Shapley et le partage égalitaire satisfont à cet axiome.

Nous pénétrons ici le cœur de l'approche axiomatique : en considérant une caractérisation donnée d'une règle d'allocation donnée, quelles sont les conséquences d'une modification, même très petite, d'un des axiomes de cette caractérisation ? Comme nous venons de le voir, la manière de traiter les joueurs nuls est centrale dans la caractérisation originelle par Shapley : n'ajoutant aucune contribution marginale à son entrée dans n'importe quelle coalition, un joueur nul reçoit un paiement nul par la propriété du joueur nul. Une des contributions majeures de cette thèse s'appuie sur d'autres variantes de cette propriété, autorisant l'implémentation de principes de justice redistributive alternatifs et menant à des règles d'allocation différentes, comme le partage égalitaire, par exemple.

Résumé des chapitres de la thèse

Cette thèse est divisée en quatre chapitres qui peuvent être lus indépendamment. Ces chapitres sont directement tirés d'articles de recherche à fins de publication et

chacun dispose de ses propres notations, rappelées dans leur introduction respective qui comporte aussi la bibliographie relative au domaine étudié. Nous sommes donc conscients que le présent résumé pourra paraître redondant à la lecture de ces introductions et nous nous efforcerons ici de mettre l'accent plutôt sur les liens entre ces différents articles, les idées-clés et les outils utilisés que sur les résultats développés en leur sein, dont nous n'énoncerons brièvement que quelques exemples. Les deux premiers chapitres de cette thèse tournent autour de l'opération de nullification, que nous définirons ci-dessous. Le troisième chapitre étudie une variante non-linéaire des valeurs de Shapley pondérées, que nous nommons valeur de Shapley proportionnelle. Dans le quatrième chapitre, nous illustrons l'approche axiomatique dans un autre cadre intéressant : l'évaluation et le classement des performances individuelles.

Le premier chapitre, publié dans *Mathematical Social Sciences* ([Béal et al., 2016](#)), utilise l'opération de nullification dans le but de modifier plusieurs axiomes relationnels populaires : l'axiome de contributions équilibrées ([Myerson, 1980](#)), l'axiome de contributions cycliquement équilibrées ([Kamijo and Kongo, 2010](#)) et l'axiome de contributions collectivement équilibrées ([Béal et al., 2016](#)). La caractéristique commune de ces axiomes est d'évaluer les conséquences du retrait d'un joueur quelconque d'un jeu TU sur le paiement d'autres joueurs. Par exemple, l'axiome de contributions équilibrées impose, pour deux joueurs quelconques, la même variation de paiement lorsque l'autre joueur quitte le jeu. Ainsi, cet axiome opère sur une classe de jeux TU dont l'ensemble des joueurs peut varier. [Myerson \(1980\)](#) montre alors qu'en combinaison avec l'axiome d'efficience, cet axiome caractérise la valeur de Shapley.

Dans notre approche alternative, au lieu de quitter le jeu, un joueur reste dans le jeu mais en devenant un joueur nul. Plus précisément, étant donné un jeu $(N, v) \in \mathbb{V}_N$ et un joueur $i \in N$, nous définissons le jeu $(N, v^i) \in \mathbb{V}_N$ tel que $v^i(S) = v(S \setminus i)$ pour tout $S \subseteq N$. Le joueur i est un joueur nul dans le nouveau jeu TU (N, v^i) : la nouvelle capacité d'une coalition le contenant est désormais égale à la capacité d'origine de la coalition sans cet agent ; les capacités des autres coalitions restant inchangées. On dira que ce joueur est nullifié dans (N, v) . On peut alors montrer que $(v^i)^i = v^i$ et que $(v^i)^j = (v^j)^i$ pour tout $j \in N$. Ainsi, pour toute coalition $S \subseteq N$, le jeu TU, noté (N, v^S) et obtenu à partir du jeu (N, v) par la nullification successive des joueurs inclus dans S (dans n'importe quel ordre), est

bien défini. Il existe de nombreux contextes économiques dans lesquels la nullification est une opération naturelle. Nous en présentons un ici (voir [Béal et al., 2016](#), pour d'autres exemples).

On suppose qu'un ensemble N de joueurs s'est mis d'accord pour covoiter sur un trajet donné pour une durée fixée de m jours (voir [Naor, 2005](#)). Pour chaque jour $k = 1, \dots, m$, notons D_k le sous-ensemble de joueurs disponibles ce jour-là et définissons ainsi le jeu TU associé : pour une coalition $S \subseteq N$, $v(S)$ est égal au nombre total de jours pour lesquels au moins un membre de S est disponible.

$$v(S) = |\{k = 1, \dots, m : D_k \cap S \neq \emptyset\}|.$$

On suppose qu'un joueur i est finalement malade et n'est plus disponible sur la durée de ces m jours. Le nouveau jeu associé (N, v') correspondant est maintenant tel que $D'_k = D_k \setminus i$ pour chaque jour k et nous avons clairement $v' = v^i$. Autrement dit, le joueur i a été nullifié dans (N, v) .

L'opération de nullification a été introduite dans [Neyman \(1989\)](#) afin de prouver l'unicité de la valeur de Shapley en appliquant au seul groupe additif engendré par un jeu TU (N, v) et tous les jeux $(N, v^S)_{S \subseteq N}$ obtenus par nullification à partir de (N, v) , les quatre axiomes susmentionnés dans la caractérisation de la valeur de Shapley ([Shubik, 1962](#)).

[Gómez-Rúa and Vidal-Puga \(2010\)](#) et [Béal et al. \(2014\)](#), dans une approche différente, mesurent l'influence de la nullification d'un joueur pour caractériser respectivement la valeur de Shapley et les partages égalitaires. La différence majeure entre le retrait d'un joueur et sa nullification tient dans le fait qu'un joueur nullifié reste encore dans le jeu, ce qui amène naturellement la question : **l'impact du retrait d'un joueur est-il équivalent à sa nullification, au sein d'un jeu TU ?** Dans ce premier chapitre, une réponse systématique à cette question est proposée en traduisant chacun des axiomes relationnels cités ci-dessus, que l'on peut appeler axiomes de « retrait », en un axiome de « nullification » correspondant. Par exemple, la version nullifiée de l'axiome des contributions équilibrées impose, pour deux joueurs quelconques, la même variation de paiement lorsque l'autre joueur est nullifié dans le jeu. Nous développons ici deux résultats intéressants pour lesquels

ce procédé de traduction a des conséquences différentes.

En premier lieu, de manière analogue à [Myerson \(1980\)](#), la combinaison de l'axiome d'efficience et des contributions équilibrées après nullification caractérise non la valeur de Shapley seule mais une famille de règles d'allocation, correspondant à des valeurs de Shapley affines, i.e. égales à la valeur de Shapley à laquelle est ajoutée un vecteur de transferts équilibré¹⁰ exogène quelconque.

Le second exemple reprend de manière similaire la caractérisation de la règle d'allocation égalitaire des coûts non séparables¹¹ EANSC par la combinaison de l'axiome d'efficience et celui des contributions collectivement équilibrées (voir [Béal et al., 2016](#)). Ce dernier impose le même impact *moyen* du retrait d'un joueur sur le paiement des joueurs restants. Notre résultat établit alors qu'il existe bien une unique règle d'allocation satisfaisant simultanément l'axiome d'efficience, celui des contributions collectivement équilibrées après nullification et le traitement égalitaire des égaux (qui est ici nécessaire à la caractérisation). Cette règle admet une formule explicite mais, contrairement à la règle d'allocation égalitaire des coûts non séparables, qui possède une saveur égalitaire, se trouve être extrêmement marginaliste : elle dote les joueurs productifs au-delà du paiement de la valeur de Shapley, au détriment des joueurs nuls qui compensent leur paiement.

Ce premier chapitre établit aussi un parallèle avec l'approche par le potentiel ([Hart and Mas-Colell, 1989](#)) et démontre que le potentiel originel, qui s'appuie sur l'opération de retrait d'un joueur, est égal à sa version nullifiée. Ainsi, de manière similaire à la caractérisation de [Hart and Mas-Colell \(1989\)](#), la valeur de Shapley pour un joueur dans un jeu TU est égale à la différence entre le potentiel « nulifié » de ce jeu TU et celui du jeu obtenu après nullification de ce joueur.

Le deuxième chapitre de cette thèse étudie un nouvel axiome sur l'ensemble des jeux TU dans lesquels l'ensemble des joueurs est fixé. Nous l'appelons **propriété**

¹⁰Un vecteur de transferts équilibré $a \in \mathbb{R}^N$ est tel que $\sum_{i \in N} a_i = 0$.

¹¹Cette règle est définie par (voir [Moulin, 1987](#), par exemple) :

$$\text{EANSC}_i(N, v) = v(N) - v(N \setminus i) + \frac{v(N) - \sum_{j \in N} (v(N) - v(N \setminus j))}{n}.$$

d'égle perte par nullification. Cet axiome utilise l'opération de nullification : il impose que lorsqu'un joueur est nullifié, tous les autres joueurs subissent la même variation de paiement. Cet axiome intègre donc une certaine solidarité entre les joueurs au travers de leur paiement et l'article étudie cet aspect selon deux approches différentes.

La première partie étudie, de façon systématique, la combinaison de l'axiome considéré avec l'axiome d'efficience. Cette analyse suggère que la propriété d'égle perte par nullification capture un trait essentiel de la classe des règles d'allocation égalitaires, comme le partage égalitaire ou le partage égalitaire du surplus (par opposition aux règles d'allocation marginalistes étudiées dans le premier chapitre). En effet, une formule générale est établie pour décrire la classe des règles d'allocation qui satisfont l'axiome d'efficience et la propriété d'égle perte par nullification ; une telle règle φ appliquée à un jeu 0-normalisé¹² quelconque coïncide avec le partage égalitaire appliqué à ce jeu, à un vecteur de transferts équilibré exogène près. Exhibons ici, pour une fois, ce résultat sous sa forme mathématisée : une règle d'allocation φ sur \mathbb{V} satisfait l'axiome d'efficience et la propriété d'égle perte par nullification si et seulement s'il existe n fonctions réelles $(F_i)_{i \in N}$ et n nombres réels $(a_i)_{i \in N}$ tels que $\sum_{i \in N} a_i = 0$, $F_i(0) = 0$ pour tout $i \in N$ et :

$$\varphi_i(v) = a_i + F_i(v(i)) - \frac{1}{n-1} \sum_{j \in N \setminus i} F_j(v(j)) + \frac{v(N)}{n}.$$

Comme cela a été dit, cette formule se simplifie en $\varphi_i(v^0) - a_i = \text{ED}_i(v^0)$ pour tout joueur $i \in N$. Cette formule générale est ensuite utilisée pour caractériser plusieurs (classes de) règles d'allocation connues : par exemple, la classe des combinaisons linéaires du partage égalitaire du surplus et du partage égalitaire correspond à la classe des règles d'allocation linéaires, symétriques et efficaces satisfaisant la propriété d'égle perte par nullification. De même, le partage égalitaire du surplus¹³ est caractérisé par l'axiome d'efficience, la propriété d'égle perte par nullification et

¹²La 0-normalisation (N, v^0) d'un jeu TU $(N, v) \in V$ quelconque est définie par $v^0(S) = v(S) - \sum_{i \in S} v(i)$ pour tout $S \subseteq N$.

¹³Cette règle d'allocation est définie par :

$$\text{ESD}_i(N, v) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}.$$

la propriété des jeux inessentiels. Ce dernier axiome impose que si tous les joueurs sont des joueurs neutres dans un jeu TU, i.e. tels qu'aucune synergie émerge de leur coopération, alors chaque joueur reçoit en paiement sa capacité singleton propre.

La seconde partie présente deux illustrations faisant intervenir les règles d'allocation précédemment caractérisées dans des contextes économiques très différents. Ici, la propriété d'égale perte par nullification est uniquement invoquée comme principe général mais aucune caractérisation axiomatique n'est avancée dans ces contextes. La première illustration considère la nullification d'un joueur comme un événement aléatoire dans un modèle de négociation. Une spécification non-linéaire et hétérogène de la formule générale présentée ci-dessus permet d'incorporer pour chaque joueur son aversion au risque dans la règle de partage, tout en conservant une certaine solidarité face à l'éventualité d'une nullification d'un des membres du groupe. Les joueurs ont aussi la possibilité de couvrir le risque encouru par leur paiement aléatoire grâce à des transferts monétaires préalables. La règle de partage proposée pour ce problème de négociation en contexte incertain permet à ceux-là d'atteindre leur paiement dans le problème de négociation certain équivalent pour chacun d'eux. La seconde application a pour but d'implémenter, par un équilibre de Nash fort, l'optimum du bien-être social d'un modèle non-coopératif de partage d'un bien commun. Cet équilibre est atteint sans l'intervention d'un planificateur social ni la création d'un marché, mais grâce à une redistribution de la production totale coïncidant avec une combinaison convexe bien précise du partage égalitaire et du partage égalitaire du surplus. Cette redistribution donne à chaque joueur une partie en proportion de leur effort individuel fourni pour la production, et une partie égalitaire (comme une bourse), levée par une taxe interne sur la production totale. En un sens, ce procédé atténue la célèbre tragédie des Communs ([Hardin, 1968](#)) sans faire intervenir un second bien, comme la monnaie, pour faciliter les transferts, ni imposer un quota fixe de production à chaque joueur.

Le troisième chapitre est consacré à l'étude axiomatique d'une nouvelle règle d'allocation : la **valeur de Shapley proportionnelle** PSh. Notre point de départ est la formule pour la valeur de Shapley :

$$\text{Sh}_i(N, v) = \sum_{S \ni i} \frac{\Delta_v(S)}{s}.$$

Cette formule fait intervenir les dividendes de Harsanyi $\Delta_v(S)$ ([Harsanyi, 1959](#)) qui, grosso modo, correspondent chacun à la « capacité intrinsèque » de la coalition S , après avoir itérativement retranché toutes les capacités intrinsèques des sous-coalitions de S :

$$\Delta_v(S) = v(S) - \sum_{T \subsetneq S} \Delta_v(T).$$

Une variante classique de la valeur de Shapley (introduite par [Shapley, 1953](#)) est la valeur de Shapley pondérée qui partage, entre leurs membres, les dividendes de Harsanyi des coalitions, non de manière égalitaire comme dans la valeur de Shapley, mais en proportion de poids positifs exogènes $w = (w_i)_{i \in N}$:

$$\text{Sh}_i^w(N, v) = \sum_{S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S).$$

La valeur de Shapley proportionnelle endogénéise ces poids en les prenant proportionnels aux capacités individuelles des joueurs :

$$\text{PSh}_i(N, v) = \sum_{S \ni i} \frac{v(i)}{\sum_{j \in S} v(j)} \Delta_v(S).$$

Ces deux dernières formules étant similaires, tous les résultats énoncés sur la classe des valeurs de Shapley pondérées n'impliquant que des fonctions caractéristiques fixes (et éventuellement les sous-jeux correspondants) sont valides pour PSh. En particulier, comme dans la caractérisation de [Myerson \(1980\)](#), PSh est caractérisée par l'axiome d'efficience et un axiome des contributions équilibrées proportionnelles. De même, une adaptation du potentiel de [Hart and Mas-Colell \(1989\)](#) permet de caractériser PSh à l'aide d'un potentiel dit proportionnel. Enfin, le résultat de [Monderer et al. \(1992\)](#) s'applique : PSh est dans le coeur d'un jeu TU convexe de la classe \mathcal{C}^0 .

En effet, remarquons que PSh n'est pas défini sur l'ensemble \mathbb{V} de tous les jeux TU. Dans cet article, nous restreignons le domaine d'étude aux jeux TU (N, v) tels que les capacités individuelles $v(i)$ sont du même signe ; cette classe correspond précisément à \mathcal{C}^0 . Remarquons aussi qu'il ne peut exister de joueur nul dans \mathcal{C}^0 et que l'opération de nullification d'un joueur nous fait sortir de \mathcal{C}^0 . Afin de caractériser PSh d'une manière comparable à Sh, une opération de neutralisation est

définie par analogie : elle correspond à la perte de synergie d'un joueur, devenant un joueur neutre. Cette nouvelle opération permet alors de transcrire les axiomes de retrait en une version neutralisée et, dans le même esprit que dans les deux premiers chapitres, PSh est caractérisée par les axiomes d'efficience, la propriété des jeux inessentiels et la version neutralisée de l'axiome des contributions équilibrées proportionnelles. Ici, la propriété des jeux inessentiels précise les paiements dans la sous-classe des jeux additifs de \mathcal{C}^0 et joue un rôle comparable à l'axiome du jeu nul dans la caractérisation de la valeur de Shapley du premier chapitre. Les deux autres axiomes étendent de manière unique la règle d'allocation à l'ensemble de la classe \mathcal{C}^0 .

La classe \mathcal{C}^0 peut sembler restrictive mais elle comprend en fait plusieurs applications économiques : notamment un contexte économique de production agricole, introduite par [Shapley and Shubik \(1967\)](#), dans lequel PSh recommande une répartition particulièrement pertinente et naturelle.

Contrairement aux valeurs de Shapley pondérées génériques, PSh satisfait le traitement égalitaire des égaux mais n'est pas linéaire. Ainsi, la caractérisation classique de la valeur de Shapley doit être adaptée pour obtenir une caractérisation comparable de PSh. Les résultats principaux de cette section reposent sur une approche similaire à la caractérisation à l'aide de l'opération de neutralisation : dans un premier temps, nous considérons une certaine combinaison d'axiomes satisfaite par PSh et telle que deux règles d'allocation, égales sur une petite sous-classe de \mathcal{C}^0 et qui satisfont cette combinaison d'axiomes, sont en fait égales sur \mathcal{C}^0 . Par exemple, si deux règles d'allocations sont égales sur la classe des jeux TU quasi-additifs¹⁴ et satisfont à la fois les axiomes d'efficience, du retrait du joueur neutre et une version affaiblie de la linéarité, alors elles sont égales sur \mathcal{C}^0 . Ici, la propriété de retrait du joueur neutre est une version analogue à celle du retrait du joueur nul, discutée précédemment. L'axiome de linéarité faible n'impose la linéarité au sein de \mathcal{C}^0 que pour des paires de jeux TU ayant des capacités singletons proportionnelles. Notons que PSh et Sh satisfont ces trois axiomes et que si ces deux règles sont égales sur la classe des jeux additifs, elles diffèrent sur celle des jeux quasi-additifs. Dans un second temps, nous spécifions l'expression de la règle d'allocation sur cette

¹⁴Un jeu TU est dit additif (resp. quasi-additif) si $v(S) = \sum_{i \in S} v(i)$ pour tout $S \subseteq N$ (resp. $S \subsetneq N$).

petite sous-classe particulière. À cet effet, comme dans la caractérisation de la valeur de Shapley par [Hart and Mas-Colell \(1989\)](#), nous considérons la combinaison d'un axiome de cohérence et d'un axiome d'étalonnage. Le principe de cohérence est un principe fondamental en sciences sociales (voir [Thomson, 2011](#), pour une revue de la littérature). Un axiome de cohérence est un axiome relationnel qui s'appuie sur la stabilité d'une règle d'allocation lors de la réduction d'un jeu TU. Il impose généralement que, lorsqu'un ou plusieurs joueurs quittent le jeu avec leur paiement, le paiement des joueurs restants dans le jeu réduit n'est pas affecté. Un axiome d'étalonnage est un axiome ponctuel. Il précise généralement les paiements dans le cas de référence d'un jeu à deux joueurs. Dans notre cas, nous définissons une version affaiblie de l'axiome de cohérence, limité aux jeux TU quasi-additifs, et un axiome d'étalonnage proportionnel, tels que PSh soit caractérisé par les cinq axiomes sus-mentionnés. De plus, en remplaçant l'étalonnage proportionnel par l'étalonnage classique, nous obtenons une caractérisation comparable de la valeur de Shapley sur \mathcal{C}^0 .

Le quatrième et dernier chapitre traite d'une variante du h -index ([Hirsch, 2005](#)), appelé le **h -index itéré**, introduit dans [García-Pérez \(2009\)](#). Ces indices tentent de quantifier des phénomènes socio-économiques comme la réputation, l'influence, la productivité ou la qualité de production des chercheurs, des journaux ou des laboratoires en s'appuyant sur le nombre de citations de chacune des publications étudiées. Plus précisément, étant donné un chercheur ayant publié n articles, son h -index est égal au plus grand entier $h \leq n$ tel que h publications aient au moins h citations. Le h -index itéré consiste alors à appliquer successivement le h -index sur les $n - h$ publications restantes (les moins citées). Ce procédé permet d'obtenir un indice multi-dimensionnel et ainsi de corriger un défaut du h -index : il permet la comparaison lexicographique entre des chercheurs ayant le même h -index et donne donc un classement plus précis.

Récemment, l'analyse quantitative des citations a rapidement développé une grande variété de méthodes de classement et, comme elle est largement utilisée dans diverses décisions en ressources humaines de nos jours, il est devenu indispensable de justifier leur utilisation d'un point de vue scientifique. Une nouvelle littérature sur le sujet essaie de combler cette lacune en utilisant l'approche axiomatique (voir

[Bouyssou and Marchant, 2014](#)). Notre contribution dans la première partie de ce chapitre appartient à ce mouvement : une caractérisation axiomatique du h -index itéré est proposée et discutée à la lumière de celle du h -index. En effet, le h -index itéré est ici caractérisé par cinq axiomes. Trois axiomes explicitent le comportement d'un indice multi-dimensionnel lors d'opérations naturelles : le premier impose que, lorsque le nombre de publications et le nombre de citations sont multipliés par un même entier, chaque composante de l'indice est elle-aussi multipliée par cet entier ; le second impose qu'en ajoutant des publications faiblement citées au palmarès du chercheur, les premières composantes de l'indice restent inchangées – qui sont liées aux publications les plus citées ; le troisième impose qu'en ajoutant encore plus de citations aux publications les plus citées, cela ne modifie pas l'indice. Ces axiomes, déjà introduits dans la littérature pour les indices uni-dimensionnels, ont été étendus ici aux indices multi-dimensionnels. Comme dans le chapitre précédent, un axiome d'étalonnage – s'occupant du cas où un chercheur ayant une ou des publications n'est cité qu'une fois en tout et pour tout – et un axiome de cohérence – qui impose que la réduction de l'ensemble des publications aux publications les moins citées se traduise par la suppression des premières composantes de l'indice – permettent de caractériser le h -index itéré. L'axiome de cohérence permet ici de distinguer le h -index itéré du h -index car, dans notre contexte multi-dimensionnel, le h -index satisfait les quatre autres axiomes et viole ce dernier.

La seconde partie de ce chapitre revient à la fonction principale d'un tel indice : intrinsèquement empirique, le h -index itéré est calculé sur des données réelles afin d'évaluer des performances dans un cadre sportif. En effet, pour les jeux d'affrontement à deux joueurs/équipes, remplacer la liste des publications par la liste des matchs gagnés, et le nombre de citations pour chaque publication par le nombre de matchs gagnés par l'équipe ou le joueur ayant perdu ce match, permet de définir le h -index itéré dans ce cadre et d'obtenir un classement alternatif. Nous présentons ici des classements pour le tennis mondial, le football français et le basketball nord-américain et discutons si le h -index itéré peut être considéré comme une bonne variable proxy pour retrouver les classements officiels dans ces sports.

Chapter 1

Axiomatic characterizations under players nullification

En théorie des jeux coopératifs, de nombreuses caractérisations axiomatiques font intervenir des axiomes évaluant les conséquences du retrait d'un joueur quelconque. L'axiome des contributions équilibrées (Myerson, 1980) et celui des contributions cycliquement équilibrées (Kamijo and Kongo, 2010) en sont deux exemples bien connus. Dans ce chapitre, nous réexaminons ces caractérisations en nullifiant le joueur plutôt qu'en le retirant du jeu. La nullification d'un joueur (Béal et al., 2014) est obtenue en modifiant le jeu initial de telle sorte que ce joueur soit un joueur nul dans le nouveau jeu, c'est-à-dire de manière que la capacité d'une coalition quelconque incluant ce joueur soit identique à celle de cette coalition sans lui. Le degré avec lequel nos résultats se rapprochent des résultats originaux est lié au fait que la valeur caractérisée vérifie ou non l'axiome de retrait du joueur nul (Derks and Haller, 1999). L'approche par le potentiel (Hart and Mas-Colell, 1989) est aussi réexaminée dans le même esprit.

Authors: Sylvain Béal, Sylvain Ferrières, Éric Rémila, Philippe Solal

Status: Published in Mathematical Social Sciences ([Béal et al., 2016](#))

Abstract:

Many axiomatic characterizations of values for cooperative games invoke axioms which evaluate the consequences of removing an arbitrary player. Balanced contributions ([Myerson, 1980](#)) and balanced cycle contributions ([Kamijo and Kongo, 2010](#)) are two well-known examples of such axioms. We revisit these characterizations by nullifying a player instead of deleting her/him from a game. The nullification ([Béal et al., 2014](#)) of a player is obtained by transforming a game into a new one in which this player is a null player, *i.e.* the worth of the coalitions containing this player is now identical to that of the same coalition without this player. The degree with which our results are close to the original results in the literature is connected to the fact that the targeted value satisfies the null player out axiom ([Derks and Haller, 1999](#)). We also revisit the potential approach ([Hart and Mas-Colell, 1989](#)) similarly.

Keywords: Player nullification, balanced contributions, Shapley value, equal allocation of non-separable costs, potential.

1.1 Introduction

This article studies cooperative games with transferable utility (denoted as TU-games). A TU-game is given by a set of players and a characteristic function which associates to any subset of players the worth created by the cooperation of its members. A value assigns to each TU-game and each player an individual payoff for participating to this TU-game. The axiomatic approach is adopted here, and following [Thomson \(2012\)](#), we sort the axioms in two kinds: punctual and relational axioms. A punctual axiom applies to each TU-game separately and a relational axiom relates payoff vectors of TU-games that are connected in a certain way. This article introduces new relational axioms.

There exist several popular relational axioms among which balanced contributions ([Myerson, 1980](#)) and balanced cycle contributions ([Kamijo and Kongo, 2010](#)). The common characteristic of these two axioms is that they evaluate the consequences of removing a player from a TU-game on the payoff of some other players. For instance, balanced contributions requires, for any two players, equal allocation variation after the leave of the other player. As such the axiomatic study in [Myerson \(1980\)](#) operates on a class of TU-games with variable player sets. Together with the standard efficiency axiom, [Myerson \(1980\)](#) characterizes the Shapley value ([Shapley, 1953b](#)).

In an alternative approach, instead of leaving the game, a player stays in the game as a null player. [Haller \(1994\)](#) makes perhaps the first step in this direction. He assumes that two players enter the TU-game with the a prior agreement specifying that one of the player acts as a proxy for the other. In the associated modified TU-game, the power of both players is shifted to the proxy player, and the other player becomes null. [Haller \(1994\)](#) invokes the axiom of proxy neutrality, which imposes that the two players' joint payoff is invariant, to characterize the Banzhaf value (see also [Casajus, 2014](#)). A similar approach is considered in [Gómez-Rúa and Vidal-Puga \(2010\)](#) and [Béal et al. \(2014\)](#) where the authors measure the influence of a complete loss of productivity of a player, in the sense that the worth of any coalition containing this player is now identical to that of the same coalition without this player. This loss of productivity of a player is called his/her nullification in reference to the fact that he/she becomes a null player. The major difference with [Haller \(1994\)](#) is that the worth of the coalitions not containing the nullified player are left unchanged. In

the body of the article, we give few examples of situations in which the nullification of a player is plausible.

In both previous cases, whether a player has left the TU-game or has been nullified means somehow that the other players cannot expect anything from this player in terms of worth. In this article, we ask the following question: is the impact of deleting a player equivalent to keeping him nullified in the TU-game? [Gómez-Rúa and Vidal-Puga \(2010\)](#) obtain a first answer by considering the nullified version of the axiom of balanced contributions. The axiom requires, for any two players, equal allocation variation after the nullification of the other player. They invoke the classical axiom of symmetry in order to recover a characterization of the Shapley value. In this article, a more systematic answer to this question is provided. To do this, we revisit several relational axioms by nullifying a player instead of removing him from the TU-game, including balanced cycle contributions ([Kamijo and Kongo, 2010](#)) and balanced collective contributions axiom ([Béal et al., 2016](#)). We also extend the analysis to the potential approach ([Hart and Mas-Colell, 1989](#)). The results below are obtained.

Firstly, the combination of balanced contributions under nullification and efficiency characterizes a class of values: each such value is the sum of the Shapley value and an exogenous budget-balanced transfer scheme. Adding the classical null game axiom, *i.e.* all players get a zero payoff if the worths of all coalitions are equal to zero, we get a characterization of the Shapley value as a corollary. These results point out that symmetry in [Gómez-Rúa and Vidal-Puga \(2010\)](#) is too strong in the sense that a mild axiom in a special case as the null game axiom suffices.

Secondly, in [Béal et al. \(2016\)](#), the equal allocation of non-separable costs (see [Moulin, 1987](#), for instance) is characterized by efficiency and balanced collective contributions. The latter axiom requires the identical average impact of the withdrawal of any player from a TU-game on the remaining population. This result is not valid anymore when efficiency is combined with our new axiom of balanced collective contributions under nullification. We prove that a new value is characterized by the latter axiom in addition to equal treatment and efficiency. This value is linear and admits a closed form expression. It relies on a marginalistic principle which goes beyond the one expressed in the Shapley value by overpaying the productive players and taxing the unproductive ones, while the equal allocation of non-separable costs possesses a more egalitarian flavor. Replacing in this last result equal treatment by

the null player axiom leads to an impossibility result.

Thirdly, we define the axiom of balanced cycle contributions under nullification as a variation of the axiom of balanced cycle contributions. The latter imposes, for all orderings of the players, that the sum of the impact on each player of removing his/her predecessor is balanced with the sum of the impact on each player of removing his/her successor. The former imposes the same requirement except that the removed players are nullified. Any linear and symmetric value satisfies both axioms. [Kamijo and Kongo \(2010\)](#) characterize the Shapley value by efficiency, null player out ([Derks and Haller, 1999](#)) and balanced cycle contributions. We prove that balanced cycle contributions under nullification, efficiency and the null player axiom characterize the Shapley value on the class of TU-games containing at least one null player. In order to recover a characterization on the full domain, we envisage two ways. On the one hand, adding linearity in the previous result yields a class of values: each such value is the sum of the Shapley value and a budget-balanced transfer scheme depending on the characteristic function in a simple manner. On the other hand, adding balanced collective contributions under nullification for TU-games possessing no null player characterizes a (non-additive) value which coincides with the Shapley value as soon as a null player is present in the TU-game, and coincides with the equal allocation of non-separable costs otherwise. In a sense, for monotonic TU-games, this value rules out any solidarity in an environment where there is at least one unproductive player. Solidarity can then emerge when every player has some positive contribution to at least one coalition.

Fourthly, we introduce a notion of nullified potential, similar to the original potential but based on the discrete derivative with respect to the nullification operation instead of the removal operation. These two potentials turn out to be equal. As a consequence, we obtain a characterization of the Shapley value analogous to the original one by [Hart and Mas-Colell \(1989\)](#): the Shapley value of a player in a TU-game is equal to the discrete derivative of the nullified potential of this TU-game with respect to this player's nullification. A recursive formula of the Shapley value relying on TU-games with nullified players is also provided in a similar way as the formula given by [Owen and Maschler \(1989\)](#).

Let us mention two other facets of our approach. Firstly, our results are valid on classes of TU-games with fixed player sets, contrary to the corresponding original results in the literature. Secondly, the axiom of null player out is useful to deepen the

relationship between the aforementioned relational axioms and their corresponding version with nullified players. In presence of null player out, we prove that the two versions of the previous relational axioms are equivalent. However the existence of a value satisfying these axioms is not always guaranteed.

The closest articles in the literature are [Gómez-Rúa and Vidal-Puga \(2010\)](#) and [Béal et al. \(2014\)](#). Apart from the aforementioned result, [Gómez-Rúa and Vidal-Puga \(2010\)](#) also consider the axiom of null balanced intracoalitional contributions for the class of TU-games with a coalition structure. This axiom is similar to nullified balanced contributions, except that only pairs of players belonging to the same cell in the coalition structure are concerned. [Béal et al. \(2014\)](#) introduce the axiom of nullified solidarity, which requires that all players weakly gain together or weakly lose together after one of them has been nullified. Together with efficiency, the null game axiom and a weak axiom of fairness, [Béal et al. \(2014\)](#) characterize the equal division value. This axiom is also mobilized in [Béal et al. \(2015\)](#) in order to characterize equal and weighted division values.

The rest of the article is organized as follows. Section 2.2 provides definitions, notations and statements of the main existing results in the literature. Section 1.3 presents the nullification of a player and some properties. Section 1.4 contains the results invoking the axioms with players nullification. Section 1.5 revisits the potential approach. Section 1.6 concludes.

1.2 Basic definitions and notations

1.2.1 Cooperative games with transferable utility

Let $\mathcal{U} \subseteq \mathbb{N}$ be a fixed and infinite universe of players. Denote by U the set of all finite subsets of \mathcal{U} . A **TU-game** is a pair (N, v) where $N \in U$ and $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. A non-empty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of the coalition. For any non-empty coalition S , let s be the cardinality of S . The **sub-game** of (N, v) induced by $S \subseteq N$ is denoted by $(S, v|_S)$, where $v|_S$ is the restriction of v to 2^S . For simplicity, we write the singleton $\{i\}$ as i . Define \mathbb{V} and $\mathbb{V}(N)$ as the classes of all TU-games with a finite player set in \mathcal{U} and of all TU-games with the fixed and finite player set $N \in U$.

Player $i \in N$ is **null** in $(N, v) \in \mathbb{V}$ if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$ such that $S \ni i$. We denote by $K(N, v)$ the set of null players in (N, v) . We often use the shortcuts $K(v)$ for $K(N, v)$ and $k(v)$ for $|K(v)|$. The subset $\mathbb{V}^0(N)$ of $\mathbb{V}(N)$ will denote the subset of TU-games with at least one null player in N . Two distinct players $i, j \in N$ are **equal** in $(N, v) \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

For $(N, v), (N, w) \in \mathbb{V}$ and $c \in \mathbb{R}$, the TU-games $(N, v + w)$ and (N, cv) are given by $(v + w)(S) = v(S) + w(S)$ and $(cv)(S) = cv(S)$ for all $S \subseteq N$. The **null game** on N is the TU-game $(N, \mathbf{0}) \in \mathbb{V}$ is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For each $N \in U$ and any nonempty $S \in 2^N$, the **unanimity TU-game** induced by S is the TU-game (N, u_S) such that $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well-known that any characteristic function $v : 2^N \rightarrow \mathbb{R}$ admits a unique decomposition in terms of unanimity TU-games:

$$v = \sum_{\emptyset \subsetneq S \subseteq N} \Delta_S(v) u_S, \quad (1.1)$$

where $\Delta_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$ is called the **Harsanyi dividend** (Harsanyi, 1959) of S in TU-game (N, v) .

1.2.2 Values

A **value** on \mathbb{V} is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any $(N, v) \in \mathbb{V}$. The definition of a value on another class of TU-games is similar. We consider the following values.

The **Shapley value** (Shapley, 1953b) is given by:

$$\text{Sh}_i(N, v) = \sum_{S \subseteq N: S \ni i} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) \quad \forall (N, v) \in \mathbb{V}, \forall i \in N.$$

The **Equal Division value** is the value ED given by:

$$\text{ED}_i(N, v) = \frac{v(N)}{n} \quad \forall (N, v) \in \mathbb{V}, \forall i \in N.$$

The **Equal Allocation of Non-Separable Costs** is the value EANSC de-

defined as:

$$\text{EANSC}_i(N, v) = v(N) - v(N \setminus i) + \frac{1}{n} \left(v(N) - \sum_{j \in N} (v(N) - v(N \setminus j)) \right) \quad \forall (N, v) \in \mathbb{V}, \forall i \in N.$$

The equal allocation of non-separable costs first assigns to each player his/her marginal contribution to the grand coalition (his/her separable cost), and then splits equally the non-separable costs among the players.

1.2.3 Some axioms and existing characterizations

In this article, we invoke axioms which can be gathered into two categories according to whether they operate on a fixed player set or on variable player sets. The first category contains the following axioms.

Efficiency, E. For all $(N, v) \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Additivity, A. For all $(N, v), (N, w) \in \mathbb{V}$, $\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w)$.

Linearity, L. For all $(N, v), (N, w) \in \mathbb{V}$ and $c \in \mathbb{R}$, $\varphi(cv + w) = c\varphi(v) + \varphi(w)$.

Null game axiom, NG. For all $(N, \mathbf{0}) \in \mathbb{V}$, all $i \in N$, $\varphi_i(N, \mathbf{0}) = 0$.

Equal treatment, ET. For all $(N, v) \in \mathbb{V}$, all $i, j \in N$ equal in (N, v) , $\varphi_i(N, v) = \varphi_j(N, v)$.

Symmetry, S. For all $(N, v) \in \mathbb{V}$, all permutation $\sigma = (i_j)_{j \in N}$ on N , $\varphi_j(N, v) = \varphi_{i_j}(N, v_\sigma)$ where v_σ is defined by $v_\sigma(S) = v(\{k | i_k \in S\})$, $S \subseteq N$.

Null player, N. For all $(N, v) \in \mathbb{V}$, all $i \in K(v)$, $\varphi_i(N, v) = 0$.

It should be clear that all the aforementioned values and axioms operating on a fixed player set N can be defined/invoked on \mathbb{V} as well. Below is a list of relational axioms operating on variable player sets.

Null player out axiom, NO. (Derks and Haller, 1999) For all $(N, v) \in \mathbb{V}$, all $i \in K(v)$, all $j \in N \setminus i$, $\varphi_j(N, v) = \varphi_j(N \setminus i, v|_{N \setminus i})$.

Balanced contributions, BC. (Myerson, 1980) For all $(N, v) \in \mathbb{V}$, for all $i, j \in N$,

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v|_{N \setminus j}) = \varphi_j(N, v) - \varphi_j(N \setminus i, v|_{N \setminus i}).$$

Balanced cycle contributions, BCyC. ([Kamijo and Kongo, 2010](#)) For all $(N, v) \in \mathbb{V}$, all ordering $(i_1, \dots, i_p, \dots, i_n)$ on N ,

$$\sum_{p=1}^n \left(\varphi_{i_p}(N, v) - \varphi_{i_p}(N \setminus i_{p-1}, v|_{N \setminus i_{p-1}}) \right) = \sum_{p=1}^n \left(\varphi_{i_p}(N, v) - \varphi_{i_p}(N \setminus i_{p+1}, v|_{N \setminus i_{p+1}}) \right),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Balanced collective contributions, BCoC. ([Béal et al., 2016](#)) For all $(N, v) \in \mathbb{V}$ with $n \geq 2$, for all $i, j \in N$

$$\frac{1}{n-1} \sum_{k \in N \setminus i} \left(\varphi_k(N, v) - \varphi_k(N \setminus i, v|_{N \setminus i}) \right) = \frac{1}{n-1} \sum_{k \in N \setminus j} \left(\varphi_k(N, v) - \varphi_k(N \setminus j, v|_{N \setminus j}) \right).$$

We conclude this section by presenting some characterizations involving axioms operating on variable sets of players.

Proposition 1.1. ([Myerson, 1980](#)) *The Shapley value is the unique value on \mathbb{V} that satisfies Efficiency (E) and Balanced contributions (BC).*

Proposition 1.2. ([Kamijo and Kongo, 2010](#)) *The Shapley value is the unique value on \mathbb{V} that satisfies Efficiency (E), Null player out (NO) and Balanced cycle contributions (BCyC).*

Proposition 1.3. ([Béal et al., 2016](#)) *The equal allocation of non-separable costs is the unique value on \mathbb{V} that satisfies Efficiency (E), and Balanced collective contributions (BCoC).*

1.3 Player's nullification

The nullification of a player in a TU-game refers to the complete loss of productivity of this player. More specifically, a new TU-game is constructed from the original one, in which the worth of any coalition containing the nullified player is equal to the worth of the same coalition without the nullified player. Formally, for $(N, v) \in \mathbb{V}$ and $i \in N$, we denote by $(N, v^i) \in \mathbb{V}$ the TU-game obtained from (N, v) if player i is **nullified**: $v^i(S) = v(S \setminus i)$ for all $S \subseteq N$. As such, i is a null player in (N, v^i) . Furthermore, it holds that $(v^i)^i = v^i$ and $(v^i)^j = (v^j)^i$ for $j \in N$. Thus,

abusing notations, for any coalition $S \subseteq N$, we denote by (N, v^S) the TU-game obtained from (N, v) by the successive nullification of each player in S (in any order). For any $(N, v) \in \mathbb{V}$, note that $(N, v^N) = (N, \mathbf{0})$.

There are various situations in which the nullification of a player arises naturally. Below, we go through three of them:

- **Glove-market games** ([Apartsin and Holzman, 2003](#)). The players in N are traders on a market with m commodities. Each traders $i \in N$ holds a certain quantity $q_k^i \in \mathbb{R}_+$ of commodity $k = 1, \dots, m$. These quantities are summarized by a vector q . There is a demand only for equal quantities of each commodity (perfect complementarity), and these are valued at unit price. In the associated TU-game (N, v^q) , the worth of each coalition $S \subseteq N$ measures the best result its members can achieve by pooling their initial bundles:

$$v^q(S) = \min \left\{ \sum_{i \in S} q_k^i : k = 1, \dots, m \right\}.$$

Let q_{-i} denote vector q without trader i 's quantities. Then it is easy to check that $v^{(0, q_{-i})}(S) = v^q(S \setminus i)$ for each $S \ni i$, and that $v^{(0, q_{-i})}(S) = v^q(S)$ otherwise. This means that if trader i carelessly loses his/her initial bundles q_k^i , $k = 1, \dots, m$, then he/she becomes nullified in the resulting TU-game. In other words, $(N, v^{(0, q_{-i})})$ is the TU-game obtained from (N, v^q) after i 's nullification.

- **Carpool games** ([Naor, 2005](#)). The players in N are people who decide to form a carpool. A schedule $D = (D_1, \dots, D_m)$ specifies, for each day $k = 1, \dots, m$, the subset of people who showed on day k . In the associated TU-game (N, v^D) , the worth of each coalition $S \subseteq N$ is equal to the total number of days members of S showed up:

$$v^D(S) = |\{k = 1, \dots, m : D_k \cap S \neq \emptyset\}|.$$

Now, imagine that participant i is on sick leave, and cannot showed up anymore during the next m days. The revised schedule D' is such that, for each day $k = 1, \dots, m$, $D'_k = D_k \setminus i$. Obviously, it holds that $v^{D'}(S) = v^D(S)$ for all $S \not\ni i$. Moreover, it is easy to check that $v^{D'}(S) = v^D(S \setminus i)$ for all $S \ni i$. This means that participant i has been nullified, or, in other words that $(N, v^{D'})$ is the TU-game obtained from (N, v^D) after i 's nullification.

- **Flow games** (Kalai and Zemel, 1982). The model is described by a graph containing two distinguished nodes called the source and the sink. Each arc has a capacity and is owned by a unique and fixed player. A player can own several arcs. In the associated TU-game, the worth of a coalition of players is measured by the maximal flow-value from the source to the sink given that only the capacities of the arcs owned by its members can be used. Similarly as before, imagine that the arcs of a player are taken away from the graph due to a network reorganization. Alternatively, suppose that the capacity of each of a player's arcs drops to zero because he/she can no longer bear the cost of maintaining them. In both cases, this player has no impact on the worth generation anymore. Therefore the resulting new TU-game is obtained from the original flow TU-game after the nullification of this player.

We continue this section by pointing out properties of the nullified TU-games.

Firstly, there is a clear relationship between the Harsanyi dividends in (N, v) and (N, v^i) : $\Delta_S(v^i) = 0$ if $S \ni i$ by definitions of a null player and of the Harsanyi dividends, and $\Delta_S(v^i) = \Delta_S(v)$ if $S \not\ni i$ since $v(T) = v^i(T)$ for all $T \subseteq S$ in such a case. Similarly, for any nonempty $T \subseteq N$, one gets $\Delta_S(v^T) = 0$ if $S \cap T \neq \emptyset$ and $\Delta_S(v^T) = \Delta_S(v)$ otherwise.

Secondly, remark that if (N, v) is a simple, monotone, superadditive, or convex TU-game, then so is (N, v^i) for all $i \in N$.

Thirdly, any characteristic function can be uniquely decomposed by means of nullified TU-games in an elegant way as established by the Lemma below.

Lemma 1.1. *For any $(N, v) \in \mathbb{V}(N)$, it holds that*

$$v = \Delta_N(v)u_N + \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} v^S. \quad (1.2)$$

Proof. Fix any $(N, v) \in \mathbb{V}(N)$. By (1.1), it is enough to prove that

$$\sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} v^S = \sum_{\emptyset \subsetneq S \subsetneq N} \Delta_S(v) u_S. \quad (1.3)$$

As a start, using (1.1), we get

$$\sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} v^S = \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} \left(\sum_{T \subseteq S} \Delta_T(v^S) u_T \right).$$

Since $\Delta_T(v^S) = 0$ if $T \cap S \neq \emptyset$ and $\Delta_T(v^S) = \Delta_T(v)$ if $T \cap S = \emptyset$, the right member in the previous equality can be rewritten as

$$\sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} \left(\sum_{T \subseteq N \setminus S} \Delta_T(v) u_T \right).$$

Rearranging this expression leads to

$$\sum_{\emptyset \subsetneq T \subsetneq N} \left(- \sum_{s=1}^{n-t} (-1)^s \binom{n-t}{s} \right) \Delta_T(v) u_T.$$

By the binomial theorem, we know that

$$- \sum_{s=1}^{n-t} (-1)^s \binom{n-t}{s} = 1,$$

which yields (1.3) and completes the proof. ■

1.4 Axiomatic study

We introduce variants of the axioms operating on variable player sets and defined in section 1.2.3 (except Null player out) by using the TU-game in which a player is nullified instead of the subgame induced by the remaining players. In this sense, we rather keep this player in the TU-game, even though he/she is nullified, instead of removing him.

Balanced contributions under nullification, BCN. For all $(N, v) \in \mathbb{V}(N)$, all $i, j \in N$,

$$\varphi_i(N, v) - \varphi_i(N, v^j) = \varphi_j(N, v) - \varphi_j(N, v^i).$$

Balanced collective contributions under nullification, BCoCN. For all $(N, v) \in \mathbb{V}(N)$, all $i, j \in N$,

$$\frac{1}{n-1} \sum_{k \in N \setminus i} \left(\varphi_k(N, v) - \varphi_k(N, v^i) \right) = \frac{1}{n-1} \sum_{k \in N \setminus j} \left(\varphi_k(N, v) - \varphi_k(N, v^j) \right).$$

Balanced cycle contributions under nullification, BCyCN. For all $(N, v) \in$

$\mathbb{V}(N)$, all ordering (i_1, \dots, i_n) of N ,

$$\sum_{p=1}^n \left(\varphi_{i_p}(N, v) - \varphi_{i_p}(N, v^{i_{p+1}}) \right) = \sum_{p=1}^n \left(\varphi_{i_p}(N, v) - \varphi_{i_p}(N, v^{i_{p-1}}) \right)$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Remark that the player set is now fixed in these axioms, so that we can consider the class of TU-games $\mathbb{V}(N)$ for some fixed player set $N \in U$.

Remark 1.1. *In our approach, the Null player out axiom plays a special role. To see this, note first that, for any $(N, v) \in \mathbb{V}$ any $j \in N$, it holds that $(N \setminus j, v^j|_{N \setminus j}) = (N \setminus j, v|_{N \setminus j})$. So, for a value φ on \mathbb{V} satisfying null player out and any $i \in N \setminus j$, it is easy to check that $\varphi_i(N, v^j) = \varphi_i(N \setminus j, v|_{N \setminus j})$. In a sense, this means that a player's nullification is equivalent to this player's removal on the payoff of the other players. More precisely, if a value φ defined on \mathbb{V} satisfies Null player out, Balanced contributions is equivalent to Balanced contributions under nullification, Balanced cycle contributions is equivalent to Balanced cycle contributions under nullification, and Balanced collective contributions is equivalent to Balanced collective contributions under nullification. The latter remark assumes that such values exist (i.e. we assume that the considered value satisfies Null player out), and we show later in this article that this is not always the case.*

Now, suppose that a value is characterized on \mathbb{V} by one axiom relying on the removal of a player, denoted by A , in addition to other axioms which only involve TU-games with fixed player set. As a consequence of the previous remark, two questions can be addressed so far. If the value characterized on \mathbb{V} satisfies Null player out, one may wonder whether replacing axiom A by its corresponding version under nullification will lead to an expanded set of values on $\mathbb{V}(N)$. Similarly, if the value characterized on \mathbb{V} does not satisfy Null player out, one may wonder whether replacing axiom A by its corresponding version under nullification will give rise to new values on $\mathbb{V}(N)$. One such value is the EANSC value. To see this, remark that the combination of Efficiency and Null player out implies Null player. So, since the EANSC value satisfies Efficiency and violates Null player, it must be the case that it violates Null player out.

The rest of this section will underline interesting differences with the existing

literature, even for results in which the characterized value satisfies Null player out.

1.4.1 Balanced contributions under nullification

In order to state our first result, the following definition is needed. A **transfer scheme** on N is a vector $a \in \mathbb{R}^N$ such that $\sum_{i \in N} a_i \leq 0$. A transfer scheme a is **budget-balanced** if $\sum_{i \in N} a_i = 0$.

Proposition 1.4. *A value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**) and Balanced contributions under nullification (**BCN**) if and only if there exists a budget-balanced transfer scheme $a \in \mathbb{R}^N$ such that $\varphi_i = \text{Sh}_i + a_i$, $i \in N$.*

The family of values characterized by Proposition 1.4 is the sum of two parts. The first one is the classical Shapley value. It is endogenous in that it relies on the worths of the considered TU-game. The second one is a budget-balanced transfer scheme. It is exogenous in that it is independent of the characteristic functions (although it depends somehow on N).

Proof. (Proposition 1.4) For each budget-balanced transfer scheme $a \in \mathbb{R}^N$, define the value $\varphi^a = \text{Sh} + a$. Firstly, φ^a obviously satisfies **E** since $\sum_{i \in N} a_i = 0$. Since Sh satisfies **BC** and **NO**, by remark 1.1, it satisfies also **BCN** on $\mathbb{V}(N)$ for any $N \in U$. The constant value assigning the payoff vector a to each TU-game $(N, v) \in \mathbb{V}(N)$ trivially satisfies **BCN**, so that φ^a satisfies **BCN**.

Secondly, let $N \in U$ and $\varphi \in \mathbb{V}(N)$ satisfying **BCN** and **E**. The proof that $\varphi = \varphi^a$ for some $a \in \mathbb{R}^N$ is done by (descending) induction on the number of null players in a TU-game. For a TU-game $(N, v) \in \mathbb{V}$, recall that $K(v)$ stands for its set of null players and that $k(v) := |K(v)|$.

INITIALIZATION. If all players are null, i.e. in the null TU-game $(N, \mathbf{0})$, define $a := \varphi(N, \mathbf{0})$. Since φ satisfies **E**, we get $\sum_{i \in N} \varphi_i(N, \mathbf{0}) = 0$, and thus a is a budget-balanced transfer scheme. Furthermore, note that $\varphi^a(N, \mathbf{0}) = \text{Sh}(N, \mathbf{0}) + a = a$ as Sh satisfies **NG**. Conclude that $\varphi(N, \mathbf{0}) = \varphi^a(N, \mathbf{0})$ for some budget-balanced transfer scheme $a \in \mathbb{R}^N$ as desired.

INDUCTION HYPOTHESIS. Assume that $\varphi(N, v) = \varphi^a$ for some budget-balanced transfer scheme a for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, $0 < k \leq n$.

INDUCTION STEP. Choose any TU-game $(N, v) \in \mathbb{V}(N)$ such that $k(v) = k - 1$. Because $k(v) < n$, there exists $i \in N \setminus K(v)$ and $K(v^i) = K(v) \cup i$ so that $k(v^i) = k(v) + 1 = k$. For all $j \in K(v)$, as Sh satisfies **N**, $\text{Sh}_j(N, v) = \text{Sh}_j(N, v^i) = 0$. Moreover $v = v^j$, so that **BCN** and the induction hypothesis imply that

$$\begin{aligned} \varphi_j(N, v) &= \varphi_j(N, v^i) + \varphi_i(N, v) - \varphi_i(N, v^j) \\ &= \varphi_j(N, v^i) \\ &= \text{Sh}_j(N, v^i) + a_j \\ &= \text{Sh}_j(N, v) + a_j. \end{aligned} \tag{1.4}$$

Conclude that the assertion is proved for null players in (N, v) . Next, for $h \in N \setminus K(v^i)$, which may be an empty set in case $k(v) = n - 1$, using **BCN** for the Shapley value and the induction hypothesis, we can rewrite **BCN** as follows:

$$\begin{aligned} \varphi_h(N, v) &= \varphi_i(N, v) + \varphi_h(N, v^i) - \varphi_i(N, v^h) \\ &= \varphi_i(N, v) + \text{Sh}_h(N, v^i) + a_h - \text{Sh}_i(N, v^h) - a_i \\ &= \varphi_i(N, v) + \text{Sh}_h(N, v) - \text{Sh}_i(N, v) + a_h - a_i \end{aligned} \tag{1.5}$$

Now **E** for φ gives:

$$\varphi_i(N, v) + \sum_{j \in K(v)} \varphi_j(N, v) + \sum_{l \in N \setminus K(v^i)} \varphi_l(N, v) = v(N).$$

Using (1.4) and (1.5) in the last equality yields:

$$\varphi_i(N, v) + \sum_{j \in K(v)} (a_j + \text{Sh}_j(N, v)) + \sum_{l \in N \setminus K(v^i)} \left(\varphi_i(N, v) + \text{Sh}_l(N, v) - \text{Sh}_i(N, v) + a_l - a_i \right) = v(N).$$

Regrouping terms:

$$(n - k(v))(\varphi_i(N, v) - a_i - \text{Sh}_i(N, v)) + \sum_{m \in N} (a_m + \text{Sh}_m(N, v)) = v(N).$$

By **E** for Sh and $\sum_{m \in N} a_m = 0$ by the induction hypothesis, we finally obtain

$$(n - k(v))(\varphi_i(N, v) - a_i - \text{Sh}_i(N, v)) = 0.$$

As $k(v) < n$ we have proved $\varphi_i(N, v) = \varphi_i^a$ for all non-null players $i \in N \setminus K(v)$ too. ■

The two axioms invoked in Proposition 1.4 are obviously logically independent. Adding the Null game axiom to them yields a characterization of the Shapley value.

Proposition 1.5. *A value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**), Balanced contributions under nullification (**BCN**), and the Null game axiom (**NG**) if and only if $\varphi = \text{Sh}$.*

The proof is a corollary of Proposition 1.4. The axioms invoked in Proposition 1.5 are logically independent:

- The value φ^a , with $a_i \neq 0$ for some $i \in N$, defined in the proof of Proposition 1.4 satisfies all axioms except **NG**.
- The value φ such that $\varphi = 2\text{Sh}$ satisfies all axioms except **E**.
- The ED-value satisfies all axioms except **BCN**.

Remark 1.2. *Balanced contributions under nullification is logically independent of Balanced contributions. On the one hand, Balanced contributions under nullification does not imply Balanced contributions by Propositions 1.1 and 1.4. On the other hand, in order to show that Balanced contributions does not imply Balanced contributions under nullification, consider the value φ on \mathbb{V} such that*

$$\varphi_i(N, v) = \sum_{S \subseteq N: S \ni i} v(S) \quad \forall (N, v) \in \mathbb{V}, \forall i \in N.$$

For any two players $i, j \in N$, simple calculations show that

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v|_{N \setminus j}) = \sum_{S \subseteq N: S \ni i, j} v(S),$$

and obviously equals $\varphi_j(N, v) - \varphi_j(N \setminus i, v|_{N \setminus i})$, while

$$\varphi_i(N, v) - \varphi_i(N, v^j) = \sum_{S \subseteq N: S \ni i, j} \left(v(S) - v(S \setminus j) \right),$$

which generically differs from $\varphi_j(N, v) - \varphi_j(N, v^i)$ unless, for instance, i and j are equals in (N, v) .

1.4.2 Balanced collective contributions under nullification

The equal allocation of non-separable costs satisfies Balanced collective contributions but neither its nullified version nor Null player out (see Remark 1.1). Furthermore, the next two results imply that this axiom is not strong enough to ensure the uniqueness of a value in presence of Efficiency. We start by an impossibility result if Null player is invoked in combination with Balanced collective contributions under nullification and Efficiency.

Proposition 1.6. *Fixed any $N \in U$ such that $n \geq 3$. There exists no value on $\mathbb{V}(N)$ satisfying Efficiency (**E**), Balanced collective contributions under nullification (**BCoCN**), and Null player (**N**).*

Proof. Consider any value φ on $\mathbb{V}(N)$ satisfying **E**, **BCoCN** and **N**. By **BCoCN**, for a given $(N, v) \in \mathbb{V}(N)$, the following sum does not depend on i and may be rewritten:

$$\sum_{k \in N \setminus i} \left(\varphi_k(N, v) - \varphi_k(N, v^i) \right) = \sum_{k \in N} \varphi_k(N, v) - \sum_{k \in N} \varphi_k(N, v^i) + \varphi_i(N, v^i) - \varphi_i(N, v).$$

By **N**, we have $\varphi_i(N, v^i) = 0$. Together with **E**, we can simplify the previous sum as $v(N) - v(N \setminus i) - \varphi_i(N, v)$ with $v^i(N) = v(N \setminus i)$. This last expression should not depend on i . Hence, it is equal to its average on N : $v(N) - (\sum_{j \in N} (v(N \setminus j)) + v(N)) / n$. It follows that:

$$\varphi_i(N, v) = \frac{1}{n} \left(v(N) + \sum_{j \in N} v(N \setminus j) \right) - v(N \setminus i) = \text{EANSC}_i(N, v).$$

Since EANSC does not satisfy **N**, we get the desired contradiction. ■

By Proposition 1.3 and the fact that the EANSC value violates Null player (see Remark 1.1), an analogue of Proposition 1.6 is obtained by replacing Balanced collective contributions under nullification by Balanced collective contributions: there exists no value on $\mathbb{V}(N)$ satisfying Efficiency, Balanced collective contributions, and Null player.

Proposition 1.6 reveals that Null player is a too strong requirement in combination with Efficiency and Balanced collective contributions under nullification. In order to circumvent this impossibility result, we replace in Proposition 1.6 the Null

player axiom by Equal treatment (i.e. null player are still treated equally, but can obtain non-null payoffs). This leads to the characterization of a new value that can be written in a closed form expression.

Proposition 1.7. *There is a unique value on $\mathbb{V}(N)$ that satisfies Efficiency (**E**), Equal treatment (**ET**) and Balanced collective contributions under nullification (**BCoCN**) which is given by*

$$SV_i(N, v) = v(N) - \frac{n-1}{n} \Delta_N(v) - \sum_{S \not\ni i} \frac{n-1}{n-s} \Delta_S(v) \quad \forall (N, v) \in \mathbb{V}, \forall i \in N. \quad (1.6)$$

Proof. Firstly, we prove uniqueness by (descending) induction on the number of null players in a TU-game. Let φ be a value on $\mathbb{V}(N)$ satisfying the three aforementioned axioms.

INITIALIZATION. If all players are null (and so equals), i.e. in the null TU-game $(N, \mathbf{0})$, **ET** and **E** implies $\varphi_i(N, \mathbf{0}) = 0$.

INDUCTION HYPOTHESIS. Assume that $\varphi(N, v)$ is uniquely determined for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, $0 < k \leq n$.

INDUCTION STEP. Choose any TU-game $(N, v) \in \mathbb{V}(N)$ such that $k(v) = k - 1$. For all $i \in N$, **E** gives:

$$\sum_{k \in N \setminus i} \left(\varphi_k(N, v) - \varphi_k(N, v^i) \right) = v(N) - \varphi_i(N, v) - v(N \setminus i) + \varphi_i(N, v^i).$$

Now **BCoCN** imposes that this last quantity should not depend on $i \in N$. If i is a null player in (N, v) , then $v = v^i$ and $v(N) = v(N \setminus i)$, so that this quantity vanishes. Two cases are to be distinguished.

For $k(v) \geq 1$, there exists at least one null player $h \in K(v)$ and we get in particular for all non-null players $i \in N \setminus K(v)$ that $\varphi_i(N, v) = v(N) - v(N \setminus i) + \varphi_i(N, v^i)$ which is uniquely determined because $k(v^i) = k(v) + 1 = k$ and by the induction hypothesis. Then **ET** applied to null players (which are equals) and **E** allows to complete the proof of uniqueness: for $h \in K(v)$, we have $\varphi_h(N, v) = (v(N) - \sum_{i \notin K(v)} \varphi_i(N, v)) / k(v)$.

For $k(v) = 0$, **BCoCN** and **E** can be used to generate a system of n linearly independent equations involving $\varphi_i(N, v)$ as the unknown variables to be expressed in terms of v and also $\varphi_i(N, v^i)$ which are determined, by induction hypothesis and

$k(v^i) = k(v) + 1 = k$ for all $i \in N$. For instance, for $N = \{1, \dots, n\}$:

$$\begin{cases} \varphi_1(N, v) - \varphi_2(N, v) & = v(N \setminus 2) - v(N \setminus 1) + \varphi_1(N, v^1) - \varphi_2(N, v^2) \\ \vdots & \ddots & = & \dots \\ \varphi_1(N, v) & - \varphi_n(N, v) & = v(N \setminus n) - v(N \setminus 1) + \varphi_1(N, v^1) - \varphi_n(N, v^n) \\ \varphi_1(N, v) + \dots + \varphi_n(N, v) & = v(N) \end{cases}$$

This implies that in this case too $\varphi_i(N, v)$ is uniquely determined (if it exists).

Secondly, we prove that SV satisfies **E**, **ET** and **BCoCN**. First notice that SV satisfies **L** so that we may use the unanimity TU-games basis for the proof. For each $N \in U$ and any nonempty $S \in 2^N$, denote by (N, δ_S) the Dirac TU-game induced by S , i.e. $\delta_S(T) = 1$ if $T = S$ and $\delta_S(T) = 0$ otherwise. For any nonempty $T \subseteq N$, we have:

$$\sum_{i \in N} \text{SV}_i(N, u_T) = n - (n-1)\delta_N(T) - \sum_{S \not\subseteq N} \sum_{i \in N \setminus S} \frac{n-1}{n-s} \delta_S(T) = n - (n-1) = 1,$$

so that **E** is proved by linearity. Next, if $i \in N$ and $j \in N$ are equal in a TU-game $(N, v) \in \mathbb{V}(N)$, recall that for all $S \subseteq N \setminus \{i, j\}$, we have $\Delta_{S \cup i}(v) = \Delta_{S \cup j}(v)$. A straight computation gives:

$$\begin{aligned} \text{SV}_i(N, v) - \text{SV}_j(N, v) &= \sum_{S \not\ni j} \frac{n-1}{n-s} \Delta_S(v) - \sum_{S \not\ni i} \frac{n-1}{n-s} \Delta_S(v) \\ &= \sum_{S \not\ni j, S \ni i} \frac{n-1}{n-s} \Delta_S(v) - \sum_{S \not\ni i, S \ni j} \frac{n-1}{n-s} \Delta_S(v) \\ &= \sum_{S \not\ni i, j} \frac{n-1}{n-(s+1)} \Delta_{S \cup i}(v) - \sum_{S \not\ni i, j} \frac{n-1}{n-(s+1)} \Delta_{S \cup j}(v) \\ &= 0. \end{aligned}$$

As in the proof of Proposition 1.6, the equality defining **BCoCN** is simplified by using **E**. Precisely, for a value φ on $\mathbb{V}(N)$ satisfying **E**, φ satisfies **BCoCN** if and only if for all $(N, v) \in \mathbb{V}(N)$, $-v(N) + v(N \setminus i) + \varphi_i(N, v) - \varphi_i(N, v^i)$ does not depend on $i \in N$. We know that SV satisfies **E**. Since $\Delta_S(v^i) = 0$ for $i \in S$ and $\Delta_S(v^i) = \Delta_S(v)$

otherwise, we get:

$$\begin{aligned}
 v(N \setminus i) + \text{SV}_i(N, v) - \text{SV}_i(N, v^i) &= v(N) - \frac{n-1}{n} \Delta_N(v) - \sum_{S \not\ni i} \frac{n-1}{n-s} \Delta_S(v) \\
 &\quad + \frac{n-1}{n} \Delta_N(v^i) + \sum_{S \not\ni i} \frac{n-1}{n-s} \Delta_S(v^i) \\
 &= v(N) - \frac{n-1}{n} \Delta_N(v).
 \end{aligned}$$

This last expression does not depend on i so we proved that **BCoCN** is satisfied by SV. ■

The axioms in Proposition 1.7 are logically independent:

- The value given by $\varphi = 2\text{SV}$ satisfies all axioms except **E**.
- The value given by $\varphi = \text{Sh}$ satisfies all axioms except **BCoCN**.
- The value given by:

$$\widehat{\text{SV}}_i(N, v) = v(N) - \frac{n-1}{n} \Delta_N(v) - \sum_{S \not\ni i} \frac{(n-1)i}{\sum_{j \in S} j} \Delta_S(v)$$

satisfies all axioms except **ET**. Indeed, $\widehat{\text{SV}}$ is also linear so that the proof of **E** and **BCoCN** are the same. Note that for two different players $i, j \in N$, if $p \in N \setminus \{i, j\}$, i and j are equals (as null players) in u_p . Now we have:

$$\begin{aligned}
 \widehat{\text{SV}}_i(N, u_p) - \widehat{\text{SV}}_j(N, u_p) &= \sum_{S \not\ni j} \frac{(n-1)j}{\sum_{k \in S} k} \delta_S(\{p\}) - \sum_{S \not\ni i} \frac{(n-1)i}{\sum_{k \in S} k} \delta_S(\{p\}) \\
 &= \frac{n-1}{\sum_{k \neq p} k} (j - i).
 \end{aligned} \tag{1.7}$$

Let us conclude this paragraph with five remarks and an example.

Firstly, we only needed Equal treatment for null players to prove the uniqueness in the proof of Proposition 1.7.

Secondly, SV satisfies Efficiency, Balanced collective contributions under nullification and the Null game axiom, which enables a comparison with Proposition 1.6.

Thirdly, replacing Balanced collective contributions under nullification by Balanced collective contributions yields an analogue of Proposition 1.7: the EANSC

value is the unique value on \mathbb{V} that satisfies Efficiency, Balanced collective contributions, and Equal treatment, even though the latter axiom is redundant as a consequence of Proposition 1.3.

Fourthly, SV can be interpreted, in particular in the unanimity TU-games. In (N, u_N) , all players get $1/n$ as it is the case for all values satisfies Efficiency and Equal treatment. For any nonempty $S \subsetneq N$, SV assigns in (N, u_S) a payoff of 1 to each player in S and so each other player get a payoff of $(1-s)/(n-s)$. These payoffs can be obtained from a two-stage procedure. In the first step, each “productive” player receives a payoff equal to the worth generated by the cooperation of all members of N , i.e. one unit. Due to the efficiency constraint, this means that a total amount of $s-1$ units has to be funded. The principle of the value SV is that this amount is exclusively funded by the unproductive players, i.e. those in $N \setminus S$. Each of them eventually pays $-(s-1)/(n-s)$. Note that the fraction $(s-1)/(n-s)$ is increasing in s , which means that the less the number of unproductive players, the more each has to pay. This interpretation also implies that SV can be seen as an extremely marginalistic value: it amplifies/exaggerates the consideration of the contributions of the players to coalition. In other words, SV implements a kind of elitism since the productive players receive an even better treatment than in the Shapley value, at the expense of the unproductive players who receive a worse treatment than in the Shapley value. In the case of a cost game, the elitism phenomenon turns into an overvaluation of the responsibility of the productive players as illustrated in the example below.

Fifthly, it is worth to note that Balanced collective contributions under nullification generates marginalistic effects through SV, which are the opposite of the more egalitarian results produced by Balanced collective contributions through the equal allocation of non-separable costs (see [Béal et al., 2016](#)).

Finally, we offer an example which emphasizes the difference between the SV value and the Shapley value by considering a pure liability TU-game ([Dehez and Ferey, 2013](#)). A victim suffers some damages. The players in N are considered to be co-responsible. The natural ordering on N represents the chain of damages caused by the players, and d_i stands for the additional damage due to player $i \in N$ after the damages caused sequentially by each of the $i-1$ first players. In the associated TU-game (N, v) , the worth of a coalition evaluates the total damage that its members would have caused by assuming the other players had followed a

nontortious behavior or had not been present. Since any pure liability game is a peer-group TU-game (Brânzei et al., 2002), it holds that

$$v = \sum_{i \in N} d_i u_{\{1, \dots, i\}},$$

which means that d_i is equal to the Harsanyi dividend of coalition $\{1, \dots, i\}$, for each $i \in N$, the dividends of all other coalitions being null. As a consequence, the Shapley value of player $i \in N$ has the following form

$$\text{Sh}_i(N, v) = \sum_{j \geq i} \frac{d_j}{j},$$

with the interpretation that the players share equally the damages occurring after their appearance in the chain of events. The SV value of player i is given by

$$\text{SV}_i(N, v) = \frac{d_n}{n} + \sum_{n > j \geq i} d_j - \sum_{j < i} \frac{j-1}{n-j} d_j.$$

The interpretation is as follows. Each player is considered fully responsible for the damages he/she caused by appearing in the chain of events, and so fully pays the total damage consecutive to his/her appearance. The only exception is the final damage, which is shared equally among all players. This makes sense since there is no extra damage to come. Furthermore, each player also receives a compensation from the players upstream of him/her in the chain of events. In a sense, the player is considered as a victim of the actions of his/her upstream players since no damage would have occurred without their presence. This is the reason why the compensation is increasing with the depth of the damage in the chain of events: a distant damage in the chain of events results from the presence/action of more players. Note also that player 1 is never compensated as establisher of the chain of damages.

As an illustration, for the case where $n = 7$ and player 5, we obtain

$$\text{SV}_5(N, v) = \frac{d_7}{7} + d_5 + d_6 - \frac{d_2}{5} - \frac{d_3}{2} - d_4,$$

while

$$\text{Sh}_5(N, v) = \frac{d_7}{7} + \frac{d_5}{5} + \frac{d_6}{6}.$$

1.4.3 Balanced cycle contributions under nullification

Similarly as in [Kamijo and Kongo \(2012\)](#), we begin by underlining that any linear and symmetric value satisfies Balanced cycle contributions under nullification.

Proposition 1.8. *For $n \geq 3$, if a value φ defined on $\mathbb{V}(N)$ satisfies Linearity (**L**) and Symmetry (**S**), then it also satisfies Balanced cycle contributions under nullification (**BCyCN**).*

The proof is similar to those in [Kamijo and Kongo \(2012\)](#) and is omitted. By remark 1.1, the characterization of the Shapley value in [Kamijo and Kongo \(2010, Proposition 1\)](#) by Efficiency, Null player out and Balanced cycle contributions is still valid if the latter axiom is replaced by its nullified counterpart. However, such a result is not in the spirit of our article where we work on a class of TU-game with a fixed player set. As a consequence, in order to cope with this constraint, a first attempt is to replace null player out by null player. We obtain the following result, which makes use of the class of all TU-games with player set N containing at least one null player.

Proposition 1.9. *For $n \geq 3$, a value φ on $\mathbb{V}^0(N)$ satisfies Efficiency (**E**), Balanced cycle contributions under nullification (**BCyCN**), and the Null player axiom (**N**) if and only if $\varphi = \text{Sh}$.*

Proof. First Sh satisfies **E** and **N** on $\mathbb{V}^0(N)$. Since Sh also satisfies **S** and **L**, it satisfies **BCyCN** by Proposition 1.8.

Now, let φ be a value on $\mathbb{V}^0(N)$ satisfying **E**, **BCyCN**, and **N**. The proof that $\varphi = \text{Sh}$ is done by (descending) induction on the number $k(v)$ of null players in a TU-game $(N, v) \in \mathbb{V}^0(N)$.

INITIALIZATION If all players are null, i.e. in the null TU-game $(N, \mathbf{0})$, we directly get $\varphi_i(N, \mathbf{0}) = 0 = \text{Sh}_i(N, \mathbf{0})$ by **N**. If all players except one are null in v , i.e. if $K(v) = N \setminus i$, **N** and **E** directly lead to $\varphi_j(N, v) = 0 = \text{Sh}_j(N, v)$ for all $j \in K(v)$ and $\varphi_i(N, v) = v(N) = \text{Sh}_i(N, v)$.

INDUCTION HYPOTHESIS. Assume that $\varphi(N, v) = \text{Sh}(N, v)$ for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, $1 < k \leq n - 1$.

INDUCTION STEP. Choose any TU-game $(N, v) \in \mathbb{V}^0(N)$ such that $k(v) = k - 1 > 0$. Because $k(v) < n - 1$, there exists at least two different non null players $i, j \in N \setminus K(v)$

and $k(v^i) = k(v^j) = k(v) + 1 = k$. By induction hypothesis, $\varphi(N, v^i) = \text{Sh}(N, v^i)$ and $\varphi(N, v^j) = \text{Sh}(N, v^j)$. Since $(N, v) \in \mathbb{V}^0(N)$, there exists $h \in K(v)$. Similarly as in [Kamijo and Kongo \(2010\)](#), **BCyCN** is equivalent to the axiom of Balanced 3-cycle contributions under nullification (i.e. when only cycles of length 3 are considered).¹ Therefore, $\varphi_i(N, v^j) + \varphi_j(N, v^h) + \varphi_h(N, v^i) = \varphi_i(N, v^h) + \varphi_j(N, v^i) + \varphi_h(N, v^j)$ which simplifies to $\varphi_j(N, v) - \varphi_i(N, v) = \text{Sh}_j(N, v^i) - \text{Sh}_i(N, v^j)$ by noting that $v^h = v$ since h is a null player. Now Sh satisfies **BCN** and we get $\varphi_j(N, v) - \varphi_i(N, v) = \text{Sh}_j(N, v) - \text{Sh}_i(N, v)$. Summing this equality for all $i \in N \setminus K(v)$ leads to:

$$\begin{aligned} (n - k(v)) \left(\varphi_j(N, v) - \text{Sh}_j(N, v) \right) &= \sum_{i \in N \setminus K(v)} \left(\varphi_i(N, v) - \text{Sh}_i(N, v) \right) \\ &\stackrel{\mathbf{N}}{=} \sum_{i \in N} \left(\varphi_i(N, v) - \text{Sh}_i(N, v) \right) \\ &\stackrel{\mathbf{E}}{=} 0. \end{aligned} \tag{1.8}$$

So we have $\varphi_j(N, v) = \text{Sh}_j(N, v)$ for non null players and, by **N** for null players too. ■

The axioms in Proposition 1.9 are logically independent:

- The Banzhaf value ([Banzhaf, 1965](#)) satisfies all axioms except **E**.
- The ED-value satisfies all axioms except **N**.
- The value given by $\varphi_i(N, v) = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$ for all $(N, v) \in \mathbb{V}^0(N)$ and $i \in N$ satisfies all axioms except **BCyCN**.

This result in Proposition 1.9 is partial since it does not deal with TU-games having no null players. There are two reasons for that. Firstly, Balanced cycle contributions under nullification and null player axiom have no implication when applied to such TU-games. For a TU-game (N, v) without null players, the payoffs in (N, v) cancel since they appear in both sides of the formula of Balanced cycle contributions under nullification. So, for such a TU-game and all ordering (i_1, \dots, i_n) of N , the axiom reduces to:

$$\sum_{p=1}^n \varphi_{i_p}(N, v^{i_{p+1}}) = \sum_{p=1}^n \varphi_{i_p}(N, v^{i_{p-1}}).$$

¹The proof of this statement and the proof of Proposition 1.8 are available upon request.

where $i_0 = i_n$ and $i_{n+1} = i_1$. Since all involved TU-games contain one null player, the axiom is silent on the original TU-game (N, v) . This is no longer the case when (N, v) possesses a null player $i \in N$ since $v = v^i$ enables to retain the original TU-game (N, v) in some part of the axiom. Secondly, [Kamijo and Kongo \(2010, 2012\)](#) use the elevator principle: starting from a TU-game with n players, they construct a TU-game with $n + 1$ players by adding a new null player, and then they come back to TU-games with n players by removing a player through the operation in Balanced cycle contributions. We cannot proceed in this fashion since the class of TU-games under consideration in this article has a fixed player set. More complete characterizations can be obtained at the cost of adding extra axioms. We present below two ways to do so. It is also interesting to remark that the set $\mathbb{V}^0(N)$ has an empty interior with respect to the natural topology on \mathbb{R}^{2^n-1} and so is a measure-zero set for any density measure on $\mathbb{V}(N)$.

Proposition 1.10. *For $n \geq 3$, a value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**), Balanced cycle contributions under nullification (**BCyCN**), the Null player axiom (**N**), and Linearity (**L**) if and only if there exists a budget-balanced transfer scheme $a \in \mathbb{R}^N$ such that for $(N, v) \in \mathbb{V}(N)$, $\varphi_i(N, v) = \text{Sh}_i(N, v) + a_i \Delta_N(v)$, $i \in N$.*

Proof. Firstly, for every budget-balanced transfer scheme $a \in \mathbb{R}^N$, the value $\text{Sh} + a\Delta_N$ obviously satisfies **L**. Since it coincides with Sh on $\mathbb{V}^0(N)$, it also satisfies **N**, and **BCyCN** on $\mathbb{V}^0(N)$ by Proposition 1.9 and the fact that $\Delta_N(v) = 0$ for all $(N, v) \in \mathbb{V}^0(N)$. Furthermore, by the remark preceding Proposition 1.10, **BCyCN** has no implication on $\mathbb{V}(N) \setminus \mathbb{V}^0(N)$, which means that $\text{Sh} + a\Delta_N$ satisfies **BCyCN** on $\mathbb{V}(N)$. Finally, the value satisfies **E** since $\Delta_N(v) \sum_{i \in N} a_i = 0$.

Secondly, let φ be a value on $\mathbb{V}(N)$ satisfying the four aforementioned axioms. By Proposition 1.9, φ coincides with Sh on $\mathbb{V}^0(N)$. Define $a_i = \varphi_i(N, u_N) - \text{Sh}_i(N, u_N)$. By **E**, we get $\sum_{i \in N} a_i = 0$ and, with the help of (1.2) in Lemma 1.1, for

any $(N, v) \in \mathbb{V}(N)$, it holds that:

$$\begin{aligned}
\varphi_i(N, v) &\stackrel{(1.2)}{=} \varphi_i\left(N, \Delta_N(v)u_N + \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} v^S\right) \\
&\stackrel{\mathbf{L}}{=} \Delta_N(v)\varphi_i(N, u_N) + \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} \varphi_i(N, v^S) \\
&\stackrel{\text{Prop. 1.9}}{=} \Delta_N(v)\varphi_i(N, u_N) + \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} \text{Sh}_i(N, v^S) \\
&\stackrel{\mathbf{L}}{=} \Delta_N(v)(\varphi_i(N, u_N) - \text{Sh}_i(N, u_N)) \\
&\quad + \text{Sh}_i\left(N, \Delta_N(v)u_N + \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} v^S\right) \\
&\stackrel{(1.2)}{=} a_i \Delta_N(v) + \text{Sh}_i(N, v),
\end{aligned}$$

which completes the proof. ■

The axioms in Proposition 1.10 are logically independent:

- The value given by $\varphi = 2\text{Sh}$ satisfies all axioms except **E**.
- The ED-value satisfies all axioms except **N**.
- The value characterized in Proposition 1.11 satisfies all axioms except **L**.
- The value given by $\varphi_i(N, v) = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$ for all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ satisfies all axioms except **BCyCN**.

Proposition 1.10 relies on the fact that any characteristic function can be decomposed into unanimity TU-games, all of which contain null players except the unanimity TU-game on the grand coalition. This result is comparable to Proposition 1.4, with the notable difference that the part including exogenous coefficients is independent of v in Proposition 1.4, while it depends on v through the Harsanyi dividend of the grand coalition in Proposition 1.10. In relevant classes of TU-games (see Maniquet, 2003, for instance), the Harsanyi dividend of the grand coalition is null, which means that Proposition 1.10 characterizes the Shapley value, provided that the other axioms are valid on the class under consideration.

Proposition 1.11. *For $n \geq 3$, a value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**), Balanced cycle contributions under nullification (**BCyCN**), the Null player axiom (**N**), and*

Balanced collective contributions under nullification on TU-games without null players (**BCoCN**^{*}) if and only if $\varphi(N, v) = \text{EANSC}(N, v)$ if $(N, v) \in \mathbb{V}(N) \setminus \mathbb{V}^0(N)$ and $\varphi(N, v) = \text{Sh}(N, v)$ if $(N, v) \in \mathbb{V}^0(N)$.

Proof. Firstly, the aforementioned value coincides with Sh on $\mathbb{V}^0(N)$, inherits Efficiency on $\mathbb{V}(N)$ from Sh and EANSC by Propositions 1.1 and 1.3. **BCyCN** and **N** are satisfied on $\mathbb{V}(N) \setminus \mathbb{V}^0(N)$ by the remark preceding Proposition 1.10, so that it only remains to prove that it satisfies **BCoCN**^{*}. Consider any $(N, v) \in \mathbb{V}(N) \setminus \mathbb{V}^0(N)$, and any $i \in N$, we have:

$$\begin{aligned} \sum_{k \neq i} \left(\text{EANSC}_k(N, v) - \text{Sh}_k(N, v^i) \right) &\stackrel{\mathbf{E}}{=} v(N) - \text{EANSC}_i(N, v) - v(N \setminus i) + \text{Sh}_i(N, v^i) \\ &\stackrel{\mathbf{N}}{=} -\frac{1}{n} \left(v(N) - \sum_{j \in N} \left(v(N) - v(N \setminus j) \right) \right) \end{aligned}$$

This last quantity does not depend on $i \in N$ so **BCoCN**^{*} is fulfilled.

Secondly, let φ be a value on $\mathbb{V}(N)$ satisfying the four aforementioned axioms. By Proposition 1.9, φ coincides with Sh on $\mathbb{V}^0(N)$. Next, for any $(N, v) \in \mathbb{V}(N) \setminus \mathbb{V}^0(N)$, **BCoCN**^{*} imposes that the following quantity is independent of $i \in N$, and in turn equal to its average on N :

$$\begin{aligned} \sum_{k \neq i} \left(\varphi_k(N, v) - \varphi_k(N, v^i) \right) &\stackrel{\mathbf{E}}{=} v(N) - \varphi_i(N, v) - v(N \setminus i) + \text{Sh}_i(N, v^i) \\ &\stackrel{\mathbf{N}}{=} v(N) - \varphi_i(N, v) - v(N \setminus i) \\ &\stackrel{\text{average}}{=} v(N) - \frac{v(N)}{n} - \frac{1}{n} \left(\sum_{j \in N} v(N \setminus j) \right) \end{aligned}$$

This last two equalities yield:

$$\varphi_i(N, v) = \frac{1}{n} \left(v(N) + \sum_{j \in N} v(N \setminus j) \right) - v(N \setminus i) = \text{EANSC}_i(N, v),$$

as desired. ■

The axioms in Proposition 1.11 are logically independent:

- The null value satisfies all axioms except **E**.

- The linear and symmetric value given by $\varphi = \text{SV}$ satisfies all axioms except **N** by Propositions 1.7 and 1.6.
- The Shapley value satisfies all axioms except **BCoCN***.
- For $i \in N$, the value given by $\varphi_i(N, v) = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$ for all $(N, v) \in \mathbb{V}^0(N)$ and $\varphi(N, v) = \text{EANSC}(N, v)$ if $(N, v) \in \mathbb{V}(N) \setminus \mathbb{V}^0(N)$ satisfies all axioms except **BCyCN**.

Proposition 1.11 calls upon several comments.

Firstly, it enables a comparison with the impossibility result in Proposition 1.6. Indeed, there is no value satisfying Efficiency, Null player and Balanced collective contributions under nullification. By Proposition 1.11, this is no longer the case if Balanced collective contributions under nullification is only required on the class of TU-games containing no null players.

Secondly, the value **SV** characterized in Proposition 1.7 by Efficiency, Equal treatment and Balanced collective contributions under nullification does not coincide with the equal allocation of non-separable costs obtained in Béal et al. (2016) if Balanced collective contributions under nullification is replaced by Balanced collective contributions. This difference is reduced if Balanced collective contributions under nullification is required only on the class of TU-games containing no null players since the equal allocation of non-separable costs takes part of the value characterized in Proposition 1.11.

Thirdly, Proposition 1.11 illustrates the fact that our relational axioms do not automatically lead to linear values. In particular, this result highlights a non continuous switch in the allocation process depending on the composition of the player set.

Regarding the last two propositions, which extend the Shapley value differently on $\mathbb{V}(N) \setminus \mathbb{V}^0(N)$, it is interesting to note that both characterized values satisfy Efficiency and Null player but differ on the two remaining axioms involved in the standard Shapley's characterization: the value in Proposition 1.10 satisfies Linearity but not Equal treatment, while the value in Proposition 1.11 satisfies Equal treatment but not Linearity. Another rather trivial extension of Proposition 1.9 is to impose Balanced contributions under nullification on TU-games without null players (**BCN***) which, together with Efficiency, Balanced cycle contributions under

nullification and Null player, characterizes the Shapley value on $\mathbb{V}(N)$.

1.5 Revisiting the potential approach

Following [Hart and Mas-Colell \(1989\)](#), a function $P : \mathbb{V} \rightarrow \mathbb{R}$ is called a **potential** if $P(\emptyset, v) = 0$ and, for all $(N, v) \in \mathbb{V}$,

$$\sum_{i \in N} (P(N, v) - P(N \setminus i, v|_{N \setminus i})) = v(N).$$

This condition means that the sum of the marginal contributions of the players in N with respect to P add up to the worth of grand coalition.

Proposition 1.12. ([Hart and Mas-Colell, 1989](#)) *There exists a unique potential function P . For all $(N, v) \in \mathbb{V}$ and $i \in N$, it is given by:*

$$P(N, v) - P(N \setminus i, v|_{N \setminus i}) = \text{Sh}_i(N, v), \quad (1.9)$$

and thus

$$P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S).$$

As in section 1.3, we substitute the TU-game in which a player is nullified for the subgame induced by the leave of this player. Formally, a **nullified potential** on N is a function $Q : \mathbb{V}(N) \rightarrow \mathbb{R}$ such that $Q(N, \mathbf{0}) = 0$ and

$$\sum_{i \in N} (Q(N, v) - Q(N, v^i)) = v(N).$$

Proposition 1.13. *There exists a unique nullified potential function Q . For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$, it holds that $Q(N, v) - Q(N, v^i) = \text{Sh}_i(N, v)$. Furthermore, $Q(N, v) = P(N, v)$.*

Proof. Firstly, we recall that the potential P satisfies:

$$P(N, v^i) = P(N \setminus i, v^i|_{N \setminus i}) + \text{Sh}_i(N, v^i) = P(N \setminus i, v^i|_{N \setminus i}) = P(N \setminus i, v|_{N \setminus i}). \quad (1.10)$$

Again, $Q(N, v) = P(N, v)$ will be proved by descending induction on the number $k(v)$ of null players in (N, v) .

INITIALIZATION. For $k(v) = n$, clearly $Q(N, \mathbf{0}) = 0 = P(N, \mathbf{0})$.

INDUCTION HYPOTHESIS. Assume that the result holds for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, for any $0 < k \leq n$.

INDUCTION STEP. Consider any $(N, v) \in \mathbb{V}(N)$ such that $k(v) = k - 1$. It holds that $k(v) < n$, so there exists at least one player $i \in N \setminus K(v)$. For any such non null player i , by the induction hypothesis we have $Q(N, v^i) = P(N, v^i)$. For any null player $j \in K(v)$, $v = v^j$ so $Q(N, v^j) = Q(N, v)$. The definition of Q and the two previous remarks then imply:

$$nQ(N, v) = v(N) + \sum_{i \in N \setminus K(v)} P(N, v^i) + \sum_{j \in K(v)} Q(N, v).$$

Hence, using **E** and **N** for Sh, we have:

$$\begin{aligned} (n - k(v))Q(N, v) &= v(N) + \sum_{i \in N \setminus K(v)} P(N, v^i) \\ &\stackrel{\mathbf{E}}{=} \sum_{i \in N} \text{Sh}_i(N, v) + \sum_{i \in N \setminus K(v)} P(N, v^i) \\ &\stackrel{\mathbf{N}}{=} \sum_{i \in N \setminus K(v)} \text{Sh}_i(N, v) + \sum_{i \in N \setminus K(v)} P(N, v^i) \\ &= \sum_{i \in N \setminus K(v)} (\text{Sh}_i(N, v) + P(N, v^i)) \\ &\stackrel{(1.10)}{=} \sum_{i \in N \setminus K(v)} (\text{Sh}_i(N, v) + P(N \setminus i, v|_{N \setminus i})) \\ &\stackrel{(1.9)}{=} \sum_{i \in N \setminus K(v)} P(N, v) \\ &= (n - k(v))P(N, v) \end{aligned}$$

This completes the proof. ■

Proposition 1.13 was expected because of Proposition 1.5 and section 3 in [Hart and Mas-Colell \(1989\)](#) in which an equivalence between the potential approach and the so-called notion of preservation of differences is established, and linked to the axiom of Balanced contributions. In our framework the only novelty is that the Null game axiom is required in addition to Balanced contribution under nullification and Efficiency in order to single out the Shapley value. This axiom somehow appears in the condition that $Q(N, \mathbf{0}) = 0$, albeit in a different form.

Our variation on the potential approach is also useful to provide a recursive

formula of the Shapley value on a class of TU-games with a fixed player set. More specifically, from [Owen and Maschler \(1989\)](#), we know that for any $(N, v) \in \mathbb{V}$ and any $i \in N$,

$$\text{Sh}_i(N, v) = \frac{1}{n} (v(N) - v(N \setminus i)) + \frac{1}{n} \sum_{j \in N \setminus i} \text{Sh}_i(N \setminus j, v|_{N \setminus j}).$$

Since the Shapley value satisfies Null player out, by [Remark 1.1](#), for all $(N, v) \in \mathbb{V}$, all $j \in N$ and all $i \in N \setminus j$, it holds that $\text{Sh}_i(N, v^j) = \text{Sh}_i(N \setminus j, v|_{N \setminus j})$. Thus, the previous expression can be rewritten as

$$\text{Sh}_i(N, v) = \frac{1}{n} (v(N) - v^i(N)) + \frac{1}{n} \sum_{j \in N \setminus i} \text{Sh}_i(N, v^j).$$

1.6 Conclusion

Our article opens the ground for an extension of the nullification approach to the class of TU-games augmented by a graph. For such TU-games, many axioms are based on deleting a link from a graph instead of removing a player. The axioms of fairness ([Myerson, 1977](#)) and component fairness ([Herings et al., 2008](#)) are two well-known examples. Instead of cutting a link, it is relevant to nullify it since this boils down to deprive this link of its ability to convey information. As such, the nullification of a link would allow to tackle the question of a network's reliability. Therefore, it would make sense to determine if small modifications of a network's reliability have the same impact on the allocation process as small alterations of the network's structure (when a link is taken away). This extension is left for a future work.

Bibliography

- Apartsin, Y., Holzman, R., 2003. The core and the bargaining set in glove-market games. *International Journal of Game Theory* 32, 189–204.
- Banzhaf, J. F., 1965. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review* 19, 317–343.
- Béal, S., Casajus, A., Huettner, F., Rémila, E., Solal, P., 2014a. Solidarity within a fixed community. *Economics Letters* 125, 440–443.
- Béal, S., Casajus, A., Huettner, F., Rémila, E., Solal, P., 2015. Characterizations of weighted and equal division values, forthcoming in *Theory & Decision*.
- Béal, S., Ferrières, S., Rémila, E., Solal, P., 2016. Axiomatic characterizations under players nullification. *Mathematical Social Sciences* 80, 47–57.
- Béal, S., Deschamps, M., Solal, P., 2016. Comparable axiomatizations of two allocation rules for cooperative games with transferable utility and their subclass of data games. *Journal of Public Economic Theory* (forthcoming).
- Brânzei, R., Fragnelli, V., Tijs, S., 2002. Tree-connected peer group situations and peer group games. *Mathematical Methods of Operations Research* 55, 93–106.
- Casajus, A., 2014. Collusion, quarrel, and the Banzhaf value. *International Journal of Game Theory* 43, 1–11.
- Dehez, P., Ferey, S., 2013. How to share joint liability: A cooperative game approach. *Mathematical Social Sciences* 66, 44–50.
- Derks, J., Haller, H. H., 1999. Null players out? Linear values for games with variable supports. *International Game Theory Review* 1, 301–314.

- Gómez-Rúa, M., Vidal-Puga, J., 2010. The axiomatic approach to three values in games with coalition structure. *European Journal of Operational Research* 207, 795–806.
- Haller, H. H., 1994. Collusion properties of values. *International Journal of Game Theory* 23, 261–281.
- Harsanyi, J. C., 1959. A bargaining model for cooperative n -person games. In: Tucker, A. W., Luce, R. D. (Eds.), *Contributions to the Theory of Games IV*. Princeton University Press, pp. 325–355.
- Hart, S., Mas-Colell, A., 1989. Potential, value, and consistency. *Econometrica* 57, 589–614.
- Herings, P. J.-J., van der Laan, G., Talman, A. J. J., 2008. The average tree solution for cycle-free graph games. *Games and Economic Behavior* 62, 77–92.
- Kalai, E., Zemel, E., 1982. Totally balanced games and games of flow. *Mathematics of Operations Research* 7, 476–478.
- Kamijo, Y., Kongo, T., 2010. Axiomatization of the Shapley value using the balanced cycle contributions property. *International Journal of Game Theory* 39, 563–571.
- Kamijo, Y., Kongo, T., 2012. Whose deletion does not affect your payoff? the difference between the Shapley value, the egalitarian value, the solidarity value, and the Banzhaf value. *European Journal of Operational Research* 216, 638–646.
- Maniquet, F., 2003. A characterization of the Shapley value in queueing problems. *Journal of Economic Theory* 109, 90–103.
- Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory* 16, 161–186.
- Myerson, R. B., 1977. Graphs and cooperation in games. *Mathematics of Operations Research* 2, 225–229.
- Myerson, R. B., 1980. Conference structures and fair allocation rules. *International Journal of Game Theory* 9, 169–182.
- Naor, M., 2005. On fairness in the carpool problem. *Journal of Algorithms* 55, 93–98.

-
- Owen, G., Maschler, M., 1989. The consistent Shapley value for hyperplane games. *International Journal of Game Theory* 18, 389–407.
- Shapley, L. S., 1953. A value for n -person games. In: *Contribution to the Theory of Games* vol. II (H.W. Kuhn and A.W. Tucker eds). *Annals of Mathematics Studies* 28. Princeton University Press, Princeton.
- Thomson, W., 2012. On the axiomatics of resource allocation: Interpreting the consistency principle. *Economics and Philosophy* 28, 385–421.

Chapter 2

Nullified equal loss property and equal division values

Nous caractérisons ici le partage égalitaire, le partage égalitaire du surplus et la classe de leurs combinaisons convexes à l'aide d'un nouvel axiome opérant sur un ensemble fixé de joueurs et faisant intervenir l'opération de nullification d'un joueur : il requiert que, lorsqu'un joueur devient nul, les allocations attribuées aux autres joueurs sont affectées de manière identique. Ce chapitre présente aussi deux applications économiques : la première concerne la négociation en contexte incertain et la seconde, le problème d'appropriation d'un bien commun.

Author: Sylvain Ferrières

Status: Under review in Theory and Decision

Abstract:

We provide characterizations of the equal division values and their convex mixtures, using a new axiom on a fixed player set based on player nullification which requires that if a player becomes null, then any two other players are equally affected. Two economic applications are also introduced concerning bargaining under risk and common-pool resource appropriation.

Keywords: Player nullification, nullified equal loss property, equal division values, bargaining under risk, common-pool resource.

2.1 Introduction

Reconciling individual and social interests is a common theme in many economics fields. For instance, solutions for bankruptcy problems often possess an egalitarian flavor (see [Thomson, 2015](#), for a recent survey). Similarly, egalitarian considerations are also central in fair division problems as pointed out by [Thomson \(2011\)](#).

Cooperative games with transferable utility (TU-games henceforth) are often used to model analogous allocation situations. A solution for a class of TU-games is called a value and assigns to each TU-game in the class and to each player a payoff for her participation. This article deals with egalitarian solutions by introducing a new axiom for TU-games called the nullified equal loss property. This axiom rests on the nullification operation studied in [Béal et al. \(2014\)](#) and [Béal et al. \(2016\)](#). A player is nullified if the worth of any coalition to which she belongs becomes equal to the worth of the same coalition without the player, i.e. the player is null in the resulting new game. The nullified equal loss property requires that if a player is nullified, then all other players experience the same payoff variation. Our results detailed in the next paragraph suggest that this axiom captures an essential feature of egalitarian values such as the equal division and equal surplus division values, as opposed to marginalistic values such as the Shapley value ([Shapley, 1953](#)). These results are in line with a recent and growing literature on the axiomatic characterizations of classes of egalitarian values ([van den Brink et al., 2016](#); [van den Brink and Funaki, 2009](#)), their axiomatic comparisons with the Shapley value ([Béal et al., 2015](#); [van den Brink, 2007](#)), and axiomatic characterizations of combination of both types of values ([Casajus and Hüttner, 2014](#); [Ju et al., 2007](#)).

The main results are as follows. Firstly, if two values satisfy the nullified equal loss property and efficiency, and furthermore coincide on the class of additive TU-games, then they are equal for all TU-games (proposition [2.1](#)). This result provides the principle of a unique extension from additive TU-games to all TU-games. Secondly, proposition [2.2](#) extends this principle for values that are linear, symmetric and efficient, and proves that the extended value must be a linear combination of the equal division value and the equal surplus division value. As a corollary, the latter class of values is characterized by linearity, symmetry, efficiency and the nullified equal loss property. Thirdly, the more natural class of convex combinations of

the equal division value and equal surplus division value is singled out by invoking efficiency, additivity, the nullified equal loss property together with desirability and superadditive monotonicity (theorem 2.1). Desirability (Maschler and Peleg, 1966) requires that if a first player contributes not less than a second player to coalitions, then the first player should not obtain a smaller payoff than the second player. Superadditive monotonicity is newly introduced and imposes that the players' payoff are nonnegative in a TU-game that is both superadditive and monotone. The axiom is implied by both monotonicity (Weber, 1988), which does not require the superadditivity of the monotone TU-game, and the axiom of nonnegativity in van den Brink et al. (2016) which imposes nonnegative payoff for nonnegative TU-games in which the grand coalition achieves a worth not less than the sum of the singletons' worth. This class emerges naturally in auction games as a mean for the player who obtains the indivisible good to compensate the other players (see van den Brink, 2007). Interestingly all axioms in theorem 2.1 except the nullified equal loss property are also satisfied by the Shapley value. This enables comparisons: replacing the nullified equal loss property by the classical null player axiom yields a characterization of the Shapley value, and replacing the nullified equal loss property by the null player in a productive environment (Casajus and Hüttner, 2013) characterizes the egalitarian Shapley values, even if some axioms may be redundant. Fourthly, thanks to proposition 2.1, an elegant characterization of the equal surplus division value is obtained by adding the well-known inessential game property to efficiency and the nullified equal loss property.

Although there are very few applications of egalitarian solutions for TU-games to economic models, the last part of this article presents two such applications. The first one considers the nullification of a player as a random event in a context of bargaining under risk. It shows that the nullified equal loss property is compatible with non-linear values that incorporate the risk aversion of the players. The second one endogeneizes a choice of a value in a non-cooperative model of common-pool management. It is shown that the unique value which maximizes the social welfare at equilibrium is a specific convex combination of equal division value and equal surplus division value.

The closest axiom to the nullified equal loss property is perhaps nullified solidarity (Béal et al., 2014). Both axioms describe the consequences of a player's nullification with two notable exception: our axiom (a) does not specify the payoff

variation for the nullified player, and (b) imposes equal payoff variation for all other players while nullified solidarity requires that all payoffs vary in the same direction. Other characterizations of the convex combinations of equal division value and equal surplus division value are due to [van den Brink et al. \(2016\)](#), while characterizations of the equal surplus division can be found in [Chun and Park \(2012\)](#) and [Béal et al. \(2015\)](#). Our results are given for fixed player sets while player sets can vary in [Chun and Park \(2012\)](#) and [van den Brink et al. \(2016\)](#). The approach by axioms of invariance in [Béal et al. \(2015\)](#) is very different from ours.

The rest of the article is organized as follows: section 2.2 presents notation and definitions. Section 2.3 contains the results. Section 2.4 presents the two applications. Section 2.5 provides concluding remarks.

2.2 Basic definitions and notations

2.2.1 Cooperative games with transferable utility

The cardinality of any set S is denoted by s . Let N be a finite and fixed set of players such that $n \geq 3$. A **TU-game** v on N is a map $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. Define \mathbb{V} as the class of all TU-games on this fixed player set N . \mathbb{V} is endowed with the natural vector space structure. A non-empty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of this coalition. For simplicity, we write the singleton $\{i\}$ as i .

The **null game** is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. A TU-game $v \in \mathbb{V}$ is **additive** if for all $S \subseteq N$, $v(S) = \sum_{i \in S} v(i)$. We will denote the class of additive TU-games by \mathbb{V}_A . For any TU-game $v \in \mathbb{V}$, let define the **0-normalized** TU-game v^0 by $v^0(S) = v(S) - \sum_{i \in S} v(i)$ for any $S \subseteq N$ so that any additive TU-game v is characterized by $v^0 = \mathbf{0}$. A TU-game $v \in \mathbb{V}$ is **superadditive** if for all $S, T \subseteq N$ such that $S \cap T = \emptyset$, $v(S \cup T) \geq v(S) + v(T)$. A TU-game $v \in \mathbb{V}$ is **monotone** if for all $S, T \subseteq N$ such that $S \subseteq T$, $v(S) \leq v(T)$. For any nonempty $S \in 2^N$, the **unanimity** TU-game induced by S is denoted by u_S and such that $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well-known that any TU-game $v \in \mathbb{V}$ admits a unique

decomposition in the unanimity games basis:

$$v = \sum_{S \in 2^N, S \neq \emptyset} \Delta_S(v) u_S,$$

where $\Delta_S(v)$ is called the Harsanyi dividend of S .

Player $i \in N$ is **null** in $v \in \mathbb{V}$ if $v(S) = v(S \setminus i)$ for all $S \subseteq N$ such that $S \ni i$. Following [Béal et al. \(2014\)](#), for $v \in \mathbb{V}$ and $i \in N$, we denote by v^i the TU-game in which player i is **nullified**: $v^i(S) = v(S \setminus i)$ for all $S \subseteq N$. Note that $(v^i)^j = (v^j)^i$ so that v^S is well-defined by nullifying all players of $S \subseteq N$, in any order. Moreover, if $S, T \subseteq N$, then $(v^S)^T = v^{S \cup T}$. For any given $v \in \mathbb{V}$, define

$$G(v) = \{v^S, S \subseteq N\}$$

the lattice generated by v using the nullification operation. Note that $v^\emptyset = v$. Moreover, $v^N = \mathbf{0}$ and $v^{N \setminus i} = v(i) \cdot u_i$ for any $i \in N$ and these TU-games are additive. At last, note that the nullification operation is compatible with the vector space structure, i.e. for all $v, w \in \mathbb{V}$, $S \subseteq N$ and $\lambda \in \mathbb{R}$, $(v + \lambda w)^S = v^S + \lambda w^S$.

2.2.2 Values

A **value** on \mathbb{V} is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}$. For any player $i \in N$, $\varphi_i(v)$ represents her payoff for participating in $v \in \mathbb{V}$. We consider the following values.

The **Equal division value** is the value ED given by:

$$\text{ED}_i(v) = \frac{v(N)}{n} \quad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Equal surplus division value** is the value ESD given by:

$$\text{ESD}_i(v) = v(i) + \frac{1}{n} \left(v(N) - \sum_{j \in N} v(j) \right) \quad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Shapley value** ([Shapley, 1953](#)) is the value Sh given by:

$$\text{Sh}_i(v) = \sum_{S \ni i} \frac{\Delta_S(v)}{s} \quad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

2.2.3 Punctual and relational Axioms

In this article, we divide axioms in two categories: punctual axioms if they impose restrictions on the payoff vector of a fixed TU-game, and relational axioms if they impose a particular relation between the payoff vectors of two different but interrelated TU-games. Two new axioms are introduced (one punctual and one relational). Let us recall first classical punctual axioms.

Efficiency, E. For all $v \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Symmetry, S. For all $v \in \mathbb{V}$, all $i, j \in N$ such that $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, we have $\varphi_i(v) = \varphi_j(v)$.

Desirability, D. (Maschler and Peleg, 1966) For all $v \in \mathbb{V}$, all $i, j \in N$ such that, for all $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) \geq v(S \cup j)$, then $\varphi_i(v) \geq \varphi_j(v)$.

Inessential game property, IGP. For all additive TU-games $v \in \mathbb{V}_A$, for all $i \in N$, $\varphi_i(v) = v(i)$.

The following new axiom imposes that a player's payoff is non negative in a superadditive and monotone TU-game.

Superadditive monotonicity, SM. For any superadditive and monotone TU-game $v \in \mathbb{V}$, all $i \in N$, $\varphi_i(v) \geq 0$.

This axiom echoes monotonicity (Weber, 1988) in which a player's payoff is required to be non negative for monotonic TU-games only. While the latter is satisfied by ED, Sh but not ESD, these three values satisfy the weaker axiom **SM**.

Below is a list of relational axioms containing our main axiom, called Nullified equal loss property. It links an arbitrary TU-game v to the TU-game v^h in which a player h is nullified, by imposing that the payoff variation should affect all the other players equally, thus preserving payoff differences among them.

Nullified equal loss property, NEL. For all $v \in \mathbb{V}$, all $h \in N$, all $i, j \in N \setminus h$,

$$\varphi_i(v) - \varphi_i(v^h) = \varphi_j(v) - \varphi_j(v^h).$$

Linearity, L. φ is a linear map $\mathbb{V} \longrightarrow \mathbb{R}^N$.

Additivity, A. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

2.3 Axiomatic study

2.3.1 General formula for efficient values satisfying the Nullified equal loss property

We begin the axiomatic study by showing that the combination of Nullified equal loss property and Efficiency implies that the values only depend on $v(S)$ for $s \in \{1, n\}$, i.e. they are determined by $n + 1$ parameters out of the $2^n - 1$ given by an arbitrary $v \in \mathbb{V}$. The following lemma is central in this approach as it allows to apprehend how these two axioms work together to restrict the value and, as corollaries, two general formulas are obtained.

Lemma 2.1. *Given a TU-game $v \in \mathbb{V}$, consider two values φ and φ' on $G(v)$ satisfying Efficiency (E) and Nullified equal loss property (NEL). If they coincide on v^S for all $S \subseteq N$ such that $s \geq n - 1$, they are equal on $G(v)$.*

Proof. Remind that $n \geq 3$ throughout the article. The proof that $\varphi(v^S) = \varphi'(v^S)$ is done by (descending) induction on the cardinal s of S .

INITIALIZATION. If $s \geq n - 1$, $\varphi = \varphi'$ by hypothesis.

INDUCTION HYPOTHESIS. Assume that $\varphi(v^S) = \varphi'(v^S)$ for all $S \subseteq N$ such that $s \geq k$ for a given $k \leq n - 1$.

INDUCTION STEP. Choose any $S \subseteq N$ such that $s = k - 1$. Because $s < n - 1$, there exists at least two distinct players $h, h' \in N \setminus S$. For all $i, j \neq h$, **NEL** and the induction hypothesis imply:

$$\varphi_i(v^S) - \varphi_j(v^S) \stackrel{\text{NEL}}{=} \varphi_i(v^{S \cup h}) - \varphi_j(v^{S \cup h}) = \varphi'_i(v^{S \cup h}) - \varphi'_j(v^{S \cup h}) \stackrel{\text{NEL}}{=} \varphi'_i(v^S) - \varphi'_j(v^S) \quad (2.1)$$

Let us show that (2.1) holds for all $i, j \in N$ without making horses the same color. Indeed, (2.1) similarly holds for $i, j \neq h'$. Thanks to $n \geq 3$, with the help of an existing $l \neq h, h'$, we have $\varphi_h(v^S) - \varphi_l(v^S) = \varphi'_h(v^S) - \varphi'_l(v^S)$ and $\varphi_l(v^S) - \varphi_{h'}(v^S) = \varphi'_l(v^S) - \varphi'_{h'}(v^S)$. Summing these last two equalities brings $\varphi_h(v^S) - \varphi_{h'}(v^S) = \varphi'_h(v^S) - \varphi'_{h'}(v^S)$, and so (2.1) holds for all $i, j \in N$. Now by summing this last equality over

$j \in N$ and using **E**, one gets:

$$n \cdot \varphi_i(v^S) - v^S(N) = n \cdot \varphi'_i(v^S) - v^S(N)$$

This immediatly leads to $\varphi_i(v^S) = \varphi'_i(v^S)$ for every $i \in N$. Conclude that $\varphi = \varphi'$ on $G(v)$. \blacksquare

Remark 2.1. Note that if a value φ satisfies **NEL** on \mathbb{V} , then for all $h \in N$, the quantity $\varphi_i(v) - \varphi_i(v^h)$ is independent of $i \neq h$ and so is equal to its average when i runs through $N \setminus h$. If φ is also efficient, this leads to:

$$\varphi_i(v) - \varphi_i(v^h) = \frac{1}{n-1} [v(N) - \varphi_h(v) - (v^h(N) - \varphi_h(v^h))]. \quad (2.2)$$

We are now ready to characterize an efficient value φ satisfying **NEL** by means of a general formula.

Corollary 2.1. A value φ on \mathbb{V} satisfies the Nullified equal loss property **NEL** and Efficiency **E** if and only if:

$$\varphi_i(v) = \varphi_i(v^{N \setminus i}) - \frac{1}{n-1} \sum_{j \in N \setminus i} \left[\varphi_j(v^{N \setminus j}) - \varphi_j(v^N) - \frac{v(j)}{n} \right] + \frac{v(N) - v(i)}{n} \quad (2.3)$$

Proof. On the one hand, the right hand side of (2.3) defines a value ψ on \mathbb{V} satisfying **NEL**. For any given $v \in \mathbb{V}$, $h \in N$ and $i \in N \setminus h$, we have:

$$\begin{aligned} \psi_i(v) - \psi_i(v^h) &= \varphi_i(v^{N \setminus i}) - \varphi_i((v^h)^{N \setminus i}) - \frac{1}{n-1} \left(\sum_{j \in N \setminus i} \left[\varphi_j(v^{N \setminus j}) - \varphi_j((v^h)^{N \setminus j}) \right. \right. \\ &\quad \left. \left. - \varphi_j(v^N) + \varphi_j((v^h)^N) - \frac{v(j) - v^h(j)}{n} \right] \right) \\ &\quad + \frac{v(N) - v^h(N) - v(i) + v^h(i)}{n} \\ &= -\frac{1}{n-1} \left(\varphi_h(v^{N \setminus h}) - \varphi_h(v^N) - \frac{v(h)}{n} \right) + \frac{v(N) - v(N \setminus h)}{n} \end{aligned}$$

which is independent of $i \in N \setminus h$.

And ψ also satisfies **E**:

$$\begin{aligned}
\sum_{k \in N} \psi_k(v) &= \sum_{k \in N} \varphi_k(v^{N \setminus k}) - \sum_{k \in N} \left(\frac{1}{n-1} \sum_{j \in N \setminus k} \left[\varphi_j(v^{N \setminus j}) - \varphi_j(v^N) - \frac{v(j)}{n} \right] \right) \\
&\quad + v(N) - \sum_{k \in N} \frac{v(k)}{n} \\
&= \sum_{k \in N} \varphi_k(v^{N \setminus k}) - \frac{1}{n-1} \left(\sum_{j \in N} \sum_{k \in N \setminus j} \left[\varphi_j(v^{N \setminus j}) - \varphi_j(v^N) - \frac{v(j)}{n} \right] \right) \\
&\quad + v(N) - \sum_{k \in N} \frac{v(k)}{n} \\
&= \sum_{j \in N} \varphi_j(v^N) + v(N) = v(N).
\end{aligned}$$

On the other hand, for any given $v \in \mathbb{V}$ and $i \in N$, ψ_i and φ_i coincide obviously on $\mathbf{0} = v^N$. Moreover:

$$\begin{aligned}
\psi_i(v^{N \setminus i}) &= \varphi_i((v^{N \setminus i})^{N \setminus i}) \\
&\quad - \frac{1}{n-1} \left(\sum_{j \in N \setminus i} \left[\varphi_j((v^{N \setminus i})^{N \setminus j}) - \varphi_j((v^{N \setminus i})^N) - \frac{v^{N \setminus i}(j)}{n} \right] \right) + \frac{v^{N \setminus i}(N) - v^{N \setminus i}(i)}{n} \\
&= \varphi_i(v^{N \setminus i}).
\end{aligned}$$

Lastly, let $h \in N \setminus i$, (2.2) leads to:

$$\begin{aligned}
\varphi_i(v^{N \setminus h}) - \varphi_i(v^N) &= \frac{1}{n-1} [v^{N \setminus h}(N) - \varphi_h(v^{N \setminus h}) + \varphi_h(v^N)] \\
&= -\frac{1}{n-1} \left[\varphi_h(v^{N \setminus h}) - \varphi_h(v^N) - \frac{v(h)}{n} \right] + \frac{v(h)}{n} \\
&= \psi_i(v^{N \setminus h}) - \psi_i(v^N)
\end{aligned}$$

where the second equality results from:

$$\frac{1}{n-1} = \frac{1}{n(n-1)} + \frac{1}{n}.$$

Therefore ψ_i and φ_i coincide on $v^{N \setminus h}$ too. By lemma 2.1, $\psi = \varphi$ on $G(v)$, and so $\psi(v) = \varphi(v)$ for any $v \in \mathbb{V}$. ■

Formula (2.3) may be written in the following simpler form:

Corollary 2.2. *A value φ on \mathbb{V} satisfies the Nullified equal loss property **NEL** and Efficiency **E** if and only if it exists n functions $(F_i)_{i \in N}$ and n numbers $(a_i)_{i \in N}$ such*

that $\sum_{i \in N} a_i = 0$, $F_i(0) = 0$ for all $i \in N$ and :

$$\varphi_i(v) = a_i + F_i(v(i)) - \frac{1}{n-1} \sum_{j \in N \setminus i} F_j(v(j)) + \frac{v(N)}{n} \quad (2.4)$$

Proof. Clear: formula (2.4) is only a recoding of formula (2.3) by setting $F_i(x) = \varphi_i(x \cdot u_i) - \varphi_i(\mathbf{0}) - x/n$ for any $x \in \mathbb{R}$ and $a_i = \varphi_i(\mathbf{0})$. ■

Remark 2.2. Formula (2.4), applied to the 0-normalized TU-game v^0 , simplifies to an affine equal division value: for any TU-game $v \in \mathbb{V}$, $\varphi_i(v^0) - \varphi_i(\mathbf{0}) = v^0(N)/n = \text{ED}_i(v^0)$. As a consequence, there is no value on \mathbb{V} that satisfies **NEL**, **E** and the well-known **Null player axiom**, **N**, defined by: for all $v \in \mathbb{V}$, all null players $i \in N$ in v , $\varphi_i(v) = 0$.

The following proposition is weaker than lemma 2.1 to characterize values satisfying **NEL** and **E** but is more convenient for the forthcoming applications.

Proposition 2.1. Consider two values φ and φ' on \mathbb{V} satisfying Efficiency (**E**) and Nullified equal loss property (**NEL**). If they coincide on the class of additive TU-games \mathbb{V}_A , they are equal on \mathbb{V} .

Proof. The proof is immediate. For any $v \in \mathbb{V}$, $v^N \in \mathbb{V}_A$ and $v^{N \setminus i} \in \mathbb{V}_A$, for all $i \in N$. Then lemma 2.1 applies. ■

2.3.2 Linear symmetric and efficient values satisfying the Nullified equal loss property

Our next result extends linear symmetric and efficient values defined on the class \mathbb{V}_A of additive TU-games to an efficient value on \mathbb{V} satisfying **NEL** in a unique way. Moreover, the class of linear symmetric efficient values satisfying **NEL** on \mathbb{V} corresponds to the class of (efficient) linear combinations of ED and ESD which, by the way, is characterized.

Proposition 2.2. If ψ is a linear symmetric and efficient value only defined on \mathbb{V}_A (i.e. satisfying **L**, **S** and **E** on \mathbb{V}_A), there exists a unique value φ satisfying Efficiency **E** and the Nullified equal loss property **NEL** on \mathbb{V} such that $\varphi = \psi$ on \mathbb{V}_A . Moreover, φ is also linear and symmetric on \mathbb{V} and there is $\lambda \in \mathbb{R}$ such that $\varphi = \lambda \text{ED} + (1 - \lambda) \text{ESD}$.

Proof. The proof relies on the fact that, on \mathbb{V}_A , **S** and **L** imply **NEL**. Indeed, let ψ be a linear symmetric value defined on \mathbb{V}_A and let $v \in \mathbb{V}_A$. Thus $v = \sum_{i \in N} v(i) \cdot u_i$ and $v^h \in \mathbb{V}_A$ for any $h \in N$. More precisely, for any $h \in N$, one has $v - v^h = v(h) \cdot u_h$ so that, for any $i, j \neq h$ and $S \subseteq N \setminus \{i, j\}$, $(v - v^h)(S \cup i) = (v - v^h)(S \cup j)$. By **S**, this implies that $\psi_i(v - v^h) = \psi_j(v - v^h)$ and **L** allows to conclude that ψ satisfies **NEL**. Suppose that ψ is also efficient and let us show that $\psi_i(u_i) = \psi_j(u_j)$ for two different players $i, j \in N$: firstly, **S** and **E** imply that:

$$1 - \psi_j(u_j) \stackrel{\mathbf{E}}{=} \sum_{k \in N \setminus j} \psi_k(u_j) \stackrel{\mathbf{S}}{=} (n-1)\psi_i(u_j). \quad (2.5)$$

Then consider $u_i + u_j \in \mathbb{V}_A$. Players i and j are symmetric, so **S** implies:

$$\begin{aligned} \psi_i(u_i + u_j) &= \psi_j(u_i + u_j) \\ \stackrel{\mathbf{L}, (2.5)}{\iff} \psi_i(u_i) + \frac{1 - \psi_j(u_j)}{n-1} &= \psi_j(u_j) + \frac{1 - \psi_i(u_i)}{n-1} \\ \iff \psi_i(u_i) &= \psi_j(u_j) \end{aligned} \quad (2.6)$$

Now let us construct a value φ on \mathbb{V} which extends ψ from \mathbb{V}_A . For $k \in N$, by analogy with formula (2.4), let $a_k = \psi_k(\mathbf{0}) = 0$ and $F_k(x) = \psi_k(x \cdot u_k) - \psi_k(\mathbf{0}) - x/n = x \cdot (\psi_k(u_k) - 1/n)$. Consider now the following value φ on \mathbb{V} :

$$\begin{aligned} \varphi_i(v) &= a_i + F_i(v(i)) - \frac{1}{n-1} \sum_{j \in N \setminus i} F_j(v(j)) + \frac{v(N)}{n} \\ &= \left(\psi_i(u_i) - \frac{1}{n} \right) v(i) - \frac{1}{n-1} \sum_{j \in N \setminus i} \left[\left(\psi_j(u_j) - \frac{1}{n} \right) v(j) \right] + \frac{v(N)}{n} \end{aligned}$$

Then φ satisfies **E**, **NEL** and **L** on \mathbb{V} . Moreover, if $v(i) = v(j)$ for $i, j \in N$, then $\varphi_i(v) - \varphi_j(v) = (1 + 1/(n-1))(\psi_i(u_i) - \psi_j(u_j))v(i) = 0$ hence φ satisfies **S**. Finally, for all $i \in N$, $\varphi_i(u_i) = \psi_i(u_i)$ and, for all $j \in N \setminus i$:

$$\varphi_i(u_j) = -\frac{\psi_j(u_j) - 1/n}{n-1} + \frac{1}{n} = \frac{1 - \psi_j(u_j)}{n-1} = \psi_i(u_j)$$

so, by linearity, $\varphi = \psi$ on \mathbb{V}_A . The uniqueness of ψ 's extension from \mathbb{V}_A to \mathbb{V} is a direct consequence of proposition 2.1.

Finally, define $\lambda = n(1 - \varphi_i(u_i))/(n - 1)$, independent of a chosen $i \in N$ by formula (2.6). Then $\varphi_i(u_i) = \lambda/n + (1 - \lambda)$ for all $i \in N$ and, by formula (2.5), $\varphi_j(u_i) = \lambda/n$ for any $j \in N \setminus i$. Hence, for all $i, j \in N$, $\varphi_i(u_j) = \lambda \text{ED}_i(u_j) + (1 - \lambda) \text{ESD}_i(u_j)$. By linearity, $\varphi = \lambda \text{ED} + (1 - \lambda) \text{ESD}$ holds on \mathbb{V}_A and finally on \mathbb{V} by proposition 2.1. ■

Remark 2.3. As it appears in proposition 2.2's proof, the general formula (2.4) for efficient values satisfying **NEL** can be particularized for linear values (satisfying **L**) by assuming linearity on \mathbb{V}_A . The corresponding functions F_i are then linear and $a_i = 0$ for all $i \in N$. Likewise, symmetric values (satisfying **S**) can be generated by imposing a symmetric treatment of players on \mathbb{V}_A only. The corresponding functions F_i are then equal and $a_i = 0$ for all $i \in N$. Clearly, these assumptions are logically independent of **E** and **NEL** on \mathbb{V} and lead to simpler formulas.

Not all symmetric efficient values defined on additive TU-games satisfies **NEL** as shown in the following example.

Example 2.1. Take $n = 3$ and ψ defined on $\mathbb{V}_A = \{x_1 u_1 + x_2 u_2 + x_3 u_3, (x_1, x_2, x_3) \in \mathbb{R}^3\}$ by:

$$\psi_i(v) = x_i(x_{i+1} - x_{i-1})^2 + \frac{x_1 + x_2 + x_3}{3} - \sum_{i \in \{1,2,3\}} x_i(x_{i+1} - x_{i-1})^2$$

where $x_4 = x_1$ and $x_0 = x_3$. Then ψ is symmetric and efficient but:

$$\psi_1(v) - \psi_2(v) = (x_3^2 - x_1 x_2)(x_1 - x_2) \neq -x_1 x_2(x_1 - x_2) = \psi_1(v^3) - \psi_2(v^3)$$

so that it does not satisfy **NEL**.

Similarly, not all linear efficient values defined on additive TU-games satisfies **NEL** as shown in the following example.

Example 2.2. Take $n = 3$ and ψ defined on $\mathbb{V}_A = \{x_1 u_1 + x_2 u_2 + x_3 u_3, (x_1, x_2, x_3) \in \mathbb{R}^3\}$ by:

$$\psi_i(v) = \frac{1}{6} (3x_i + 2x_{i+1} + x_{i-1})$$

where $x_4 = x_1$ and $x_0 = x_3$. Then ψ is linear and efficient but:

$$\psi_1(v) - \psi_2(v) = \frac{1}{6} (2x_1 - x_2 - x_3) \neq \frac{1}{6} (2x_1 - x_2) = \psi_1(v^3) - \psi_2(v^3)$$

so that it does not satisfy **NEL**.

As a direct consequence of proposition 2.2, we have the following characterization.

Corollary 2.3. *A value φ on \mathbb{V} satisfies Efficiency (E), Nullified equal loss property (NEL), Linearity (L) and Symmetry (S) if and only if there is $\lambda \in \mathbb{R}$ such that $\varphi = \lambda \text{ED} + (1 - \lambda) \text{ESD}$.*

2.3.3 Characterization of the class of convex combinations of ED and ESD

By relying on the previous result, we characterize the more natural class of convex combinations of ED and ESD.

Theorem 2.1. *A value φ on \mathbb{V} satisfies Efficiency (E), Nullified equal loss property (NEL), Additivity (A), Desirability (D) and Superadditive Monotonicity (SM) if and only if there is $\lambda \in [0, 1]$ such that:*

$$\varphi = \lambda \text{ED} + (1 - \lambda) \text{ESD}.$$

Proof. For any superadditive and monotone TU-game $w \in V$, note that for all $i \in N$, $0 \leq w(i) \leq w(N)$ and $w(N) \geq \sum_{i \in N} w(i)$ so that $\text{ED}_i(w) \geq 0$ and $\text{ESD}_i(w) \geq 0$. Thus any convex combination φ of ED and ESD satisfies SM. Moreover, for any $i, j \in N$,

$$\varphi_i(v) - \varphi_j(v) = (1 - \lambda)(v(i) - v(j)) \quad (2.7)$$

so that φ satisfies D and, by corollary 2.3, φ satisfies all the other involved axioms. Reciprocally, let φ be a value satisfying the five aforementioned axioms. By Casajus and Hüttner (2013, Lemma 5), A, E and D imply L. Moreover, D implies S. By corollary 2.3, there is $\lambda \in \mathbb{R}$ such that $\varphi = \lambda \text{ED} + (1 - \lambda) \text{ESD}$. From formula (2.7) and D, we get $\lambda \leq 1$. Finally, SM applied to the superadditive and monotone TU-game u_i for a fixed $i \in N$ brings $\varphi_j(u_i) = \lambda/n \geq 0$ for all $j \in N \setminus i$. ■

The axioms invoked in Theorem 2.1 (as well as in corollary 2.3) are logically independent:

- The value $\varphi = 2\text{ED}$ satisfies all axioms except E.

- The value $\varphi = \text{Sh}$ satisfies all axioms except **NEL**, as a consequence of remark 2.2.
- Single out a player $i_0 \in N$ and define φ for all $v \in \mathbb{V}$ by:

$$\varphi_i(v) = \begin{cases} -\frac{1}{n-1} \cdot \frac{2}{3} \cdot v(i_0) + \frac{v(N)}{n} & \text{if } i \in N \setminus i_0 \\ \frac{2}{3} \cdot v(i_0) + \frac{v(N)}{n} & \text{if } i = i_0 \end{cases}$$

Then φ satisfies **E**, **A** and **NEL**. Moreover if w is a superadditive and monotone TU-game, $w(N) \geq w(i_0) \geq 0$. Because $(n-1)/n \geq 2/3$ for $n \geq 3$, $w(N)/n \geq 2w(i_0)/3(n-1)$ so that $\varphi_i(w) \geq 0$ for all $i \in N$. Hence, **SM** is also satisfied by φ . However, **S** is clearly not satisfied so that **D** is violated.

- The value $\varphi = 2\text{ESD} - \text{ED}$ satisfies all axioms except **SM**. Indeed, consider for instance the unanimity game u_i , for a given player $i \in N$, it is a superadditive and monotone TU-game and for any $j \in N \setminus i$, $\varphi_j(u_i) = -1/n < 0$.
- The value φ defined by $\varphi_i(v) = \max(0, v(i)) + (v(N) - \sum_{j \in N} \max(0, v(j)))/n$ for all $i \in N$ satisfies all axioms except **A**.

Remark 2.4. As mentioned in the introduction, theorem 2.1 can be used to compare the class of convex combinations of ED and ESD with the Shapley value and the class of egalitarian Shapley values. The latter class consists of all convex combinations of Sh and ED. It is easy to check that all the aforementioned values satisfy **SM**. Replacing **NEL** by **N** (resp. **Null player in a productive environment, NPE**).¹ introduced in Casajus and Hüttner, (2013) yields a (redundant) characterization of Sh (resp. the class of egalitarian Shapley values).

2.3.4 Punctual characterization of equal division values

This section provides characterizations of ESD and ED that only differ with respect to the requirements on additive TU-games, attesting to the centrality of **NEL** axiom in the context of equal division values.

¹For all $v \in V$ with $v(N) \geq 0$, all null players $i \in N$ in v , we have $\varphi_i(v) \geq 0$.

Proposition 2.3. *A value φ on \mathbb{V} satisfies Efficiency (**E**), Nullified equal loss property (**NEL**) and the Inessential game property (**IGP**) if and only if it is the equal surplus division value $\varphi = \text{ESD}$.*

Proof. The result is a straight consequence of proposition 2.1: **IGP** characterizes a unique value on \mathbb{V}_A and ESD satisfies **NEL**, **IGP** and **E**. By proposition 2.2, the logical independence is obvious. ■

Remark 2.5. *The equal division value can be characterized with a similar set of three axioms. For this purpose, we introduce an ad hoc axiom **Equal division for inessential games, EIG**: for all additive TU-games $v \in \mathbb{V}_A$, for all $i \in N$, $\varphi_i(v) = (\sum_{j \in N} v(j))/n$. One easily gets that **E**, **NEL** and **EIG** characterizes ED. More generally, for a fixed $\lambda \in [0; 1]$, the convex combination $\lambda \text{ED} + (1 - \lambda) \text{ESD}$ is characterized by **E**, **NEL** and λ -**IGP** where the latter axiom is defined by: for all additive TU-games $v \in \mathbb{V}_A$, for all $i \in N$, $\varphi_i(v) = \lambda v(i) + (1 - \lambda)(\sum_{j \in N} v(j))/n$. By lemma 2.1, these characterizations can be weakened by only requiring λ -**IGP** for all multiple of unanimity games $x \cdot u_i$ where $i \in N$, $x \in \mathbb{R}$.*

2.4 Applications

This section presents two applications of the values involved in the preceding sections. Our aim is not to characterize them in other axiomatic contexts and this aspect is left for future work. The first one rests on formula (2.4). This expression does not specify the shape of functions F_i and so allows to grasp situations in which non-linearity and individual specificities are important features. More specifically, we consider a situation of bargaining under risk, dealing with risk aversion, which cannot be handled with symmetric or linear values only.

2.4.1 Bargaining under risk

Most economic models of bargaining assume certainty of outcomes or risk-neutrality of negotiators although many real life situations involve pay-off uncertainty which may arise from various random events. A typical n -bargaining situation à la Nash is usually described by a pair (\mathcal{C}, d) composed of a convex, comprehensive subset $\mathcal{C} \subseteq \mathbb{R}^n$ of feasible outcomes and a disagreement point $d \in \mathbb{R}^n$. If all players

agree on a point $x \in \mathcal{C}$, they get x . Otherwise, they obtain d . A solution is a function associating with every (\mathcal{C}, d) a feasible outcome $F(\mathcal{C}, d) \in \mathcal{C}$ representing the compromise unanimously reached by the players. Here we consider a fixed set N of players in a risky bargaining situation where every player may be independently affected by an entire loss of productivity as modeled by a nullification, involving both the disagreement point and the set of feasible outcomes. Hence players may face one of the 2^n different n -bargaining situations resulting from all possible nullifications. If $S \subsetneq N$ is the set of nullified players, we limit the corresponding set of feasible outcomes to $\mathcal{C}_S = \{(x_i)_{i \in N} \in \mathbb{R}^n, \sum_{i \in N} x_i \leq W_S\}$ where $W_S > 0$ is a worth to be shared². This generates a positive TU-game $v \in \mathbb{V}$ by setting $v(N \setminus S) = v^S(N) = W_S$ and we assume that v is also superadditive. The objective is to fairly distribute the worth finally achievable by the society N among its members. Besides, players are not allowed to await the realizations of the potential nullifications before deciding upon a joint sharing scheme, i.e. they have to design a value φ on $G(v)$. At last, each player is characterized by an individual risk aversion, which alters her bargaining power accordingly and incents her to hedge her stand-alone risk by prior monetary transfers.

Formally, this situation is essentially described by three elements:

- an individual and independent probability $p_i \in]0, 1[$ that measures the risk of being nullified faced by player $i \in N$, in the sense that i is fully productive with probability $1 - p_i$ and loses her productivity with probability p_i . Note $p = (p_i)_{i \in N} \in]0, 1[^N$ the corresponding vector;
- a positive and superadditive TU-game v on N so that $v^S(N) = v(N \setminus S)$ evaluates, for any $S \subseteq N$, the worth to be shared in the bargaining situation where S is the set of nullified players;
- an individual utility function w_i for each player $i \in N$ which takes player i 's risk aversion into account. We require that these utility functions should be defined on \mathbb{R} , strictly increasing and strictly concave such that $w_i(0) = 0$. A well-known example is the CARA utility function (see for instance [Pratt, 1964](#)) of the form $w_i(x) = (1 - e^{-\alpha_i x})/\alpha_i$ for $x \in \mathbb{R}$ where $\alpha_i > 0$ is the individual constant absolute

²This framework may seem analog to hyperplane games introduced in [Maschler and Owen \(1989\)](#) but with the main difference that W_S is shared here by the whole fixed player set N and not by $N \setminus S$, as in the framework of non-sidepayment cooperative games.

risk aversion parameter. Note $w = (w_i)_{i \in N}$ so that the disagreement point d_S is $(w_i(v^S(i)))_{i \in N}$ in the bargaining situation where S is the set of nullified players.

For a situation (p, v, w) on N , define the average bargaining situation by:

$$v_p = \sum_{S \subseteq N} \prod_{j \in S} p_j \prod_{i \notin S} (1 - p_i) v^S.$$

This expression is similar to Owen's multilinear extension of TU-games (see [Owen, 1972](#)). For each coalition $S \subseteq N$, $v_p(S)$ can be considered as the average worth to be shared given that (i.e. conditionally to) $N \setminus S$ is nullified for sure.

Given an efficient value φ , a solution to the bargaining situation (\mathcal{C}_S, d_S) may be denoted by $\varphi(v^S)$. For each player $k \in N$, the average allocation is denoted by:

$$\varphi_k^p(v) = \sum_{S \subseteq N} \prod_{j \in S} p_j \prod_{i \notin S} (1 - p_i) \varphi_k(v^S).$$

Requiring efficiency for φ in this context can be seen as the risk-neutrality of the grand coalition:

$$\sum_{i \in N} \varphi_i(v_p) = v_p(N) = \sum_{i \in N} \varphi_i^p(v). \quad (2.8)$$

The following axiom is a collective variant of our axiom **NEL** and is defined by:

Group-Nullified equal loss property, GNEL. For all $v \in \mathbb{V}$, all $S \subseteq N$, all $i, j \in N \setminus S$,

$$\varphi_i(v) - \varphi_i(v^S) = \varphi_j(v) - \varphi_j(v^S).$$

The **GNEL** axiom is interpreted similarly to **NEL**: bargaining players in $N \setminus S$ incur the same difference in payoff when coalition S becomes nullified. This axiom is a natural requirement for φ in this context of risk hedging and turns out to be equivalent to **NEL** (by successive application). Thus values on $G(v)$ compatible with **GNEL** and **E** are given by formula (2.4). An example of such a value can be obtained by setting for instance $a_i = 0$ and $F_i(x) = (1 - 1/n) \cdot w_i(x)$. This particular value can be naturally extended from $G(v)$ to \mathbb{V} so that $\varphi(v_p)$ may be computed.

After simplifications, this brings:

$$\varphi_i(v) = w_i(v(i)) + \frac{v(N) - \sum_{j \in N} w_j(v(j))}{n}.$$

The allocation $\varphi(v^S)$ in our context corresponds to the egalitarian solution (Kalai, 1977) applied on the bargaining situation (\mathcal{C}_S, d_S) . Remark also that, in the particular case where the utility functions are CARA, $w_i(v(i))$ tends to $v(i)$ when α_i tends to 0 so that, when all players are risk-neutral, the disagreement point is $d = (v(i))_{i \in N}$. In this case, we recover ESD as the egalitarian solution when utility is transferable. Let us emphasize the following fact: when players become risk-adverse, with possibly different individual risk-aversion parameters, only the disagreement points d_S are impacted. This is consistent with the *individual* aspect of these outcomes and we do not exclude that d_S may be outside \mathcal{C}_S . However, if an agreement is found, the worth to be shared is transferable, independent of individual utilities and only depends, in our context, on the set of nullified players. Bearing this idea in mind, the players have the possibility of making transfers prior to the actual realization of the potential nullifications. We obtain the following result:

Result 2.1. *There exists a unique budget-balanced transfer scheme π of risk premia between the players so that any player's average allocation equals her allocation in the non-random (or certainty equivalent) bargaining situation³ $v_p - \pi$:*

$$\varphi_i(v_p - \pi) = \varphi_i^p(v) \text{ for all } i \in N.$$

Proof. Firstly, φ is efficient. Equation (2.8) implies that $\sum_{i \in N} \pi_i = 0$, so that the aforesaid transfer scheme π is budget-balanced. Secondly, it is easy to show that:

$$\begin{aligned} \varphi_i^p(v) &= (1 - p_i)w_i(v(i)) + \frac{v_p(N) - \sum_{j \in N} (1 - p_j)w_j(v(j))}{n} \\ \varphi_i(v_p - \pi) &= w_i((1 - p_i)v(i) - \pi_i) + \frac{v_p(N) - \sum_{j \in N} w_j((1 - p_j)v(j) - \pi_j)}{n}. \end{aligned}$$

The equation $\varphi_i(v_p - \pi) = \varphi_i^p(v)$ then becomes:

$$T_i(\pi_i) = \frac{1}{n} \sum_{j \in N} T_j(\pi_j) \tag{2.9}$$

³Abusing notation, π also denotes the induced additive TU-game.

where $T_i(x) = w_i((1 - p_i)v(i) - x) - (1 - p_i)w_i(v(i))$, for all $i \in N$. The function T_i is strictly decreasing. Denote by $q_i = \lim_{-\infty} T_i > 0$ which exists and may be infinite. Remark that $\lim_{+\infty} T_i = -\infty$. Hence T_i is a continuous strictly decreasing bijection between \mathbb{R} and $] -\infty, q_i[$. Note also that $T_i(0) > 0$ whenever $v(i)(1 - p_i) \neq 0$ by strict concavity.

The system (2.9) of n linear equations in $t_i = T_i(\pi_i)$ is underdetermined of rank $n - 1$. Indeed, the solutions are parametrized by $t \in \mathbb{R}$ such that $t_i = t$ for all $i \in N$. Define $q = \min_{i \in N} q_i > 0$ (which may be infinite) and consider the continuous strictly decreasing function $Q(x) = \sum_{i \in N} T_i^{-1}(x)$ defined on $] -\infty, q[$. One has $\lim_{-\infty} Q = +\infty$ and $\lim_{q^-} Q = -\infty$ so there exists a unique $t^* \in] -\infty, q[$ such that $Q(t^*) = 0$. Finally the transfer scheme defined by $\pi_i = T_i^{-1}(t^*)$ satisfies all desired conditions and depends on p , w and the stand-alone capacities $(v(i))_{i \in N}$ only. ■

In our context, transfers only result from the non-linearity of φ , through the non-linearity of w . Indeed, for linear values, we have $\varphi_i(v_p) = \varphi_i^p(v)$ unconditionally. Likewise, if $T_i = T$ for all players $i \in N$ (for instance when all individual variables $v(i)$, p_i and w_i are equal), i.e. in a symmetric framework, then $\pi_i = 0$ and no transfer is needed. Moreover, when there is no random effect, i.e. if $p_i = 0$ (resp. $p_i = 1$) for all $i \in N$, one may also show that $t = 0$ and $\pi_i = 0$ for all $i \in N$.

2.4.2 Softening the tragedy of the Commons

The second application illustrates the interest of convex combinations of equal division values in a well-known economic context. Consider a perfectly divisible common-pool resource (CPR) for which no storage is feasible and operated by a fixed community N of potential consumers, facing pure appropriation externalities (see [Ostrom et al., 1994](#), for a wide overview). In this context, uncoordinated individual consumption leads the aggregated society to deviate from an optimal social welfare. Suppose that the socially optimal overall consumption is independent of how this consumption is divided among the players. If, for any reason, any player does not to consume the CPR, this will not affect the community's consumption optimum but, in a symmetric framework, other players will have to equally compensate this gap so that the community's consumption remains optimal. This last comment allows an analogy with the **NEL** principle.

Let us now present a model close to [Funaki and Yamato \(1999\)](#). Suppose that a constant and common marginal labor cost $q > 0$ is needed to exploit the CPR and denote by $x = (x_i)_{i \in N}$ the vector of individual work efforts so that $x_i \in \mathbb{R}^+$ for player i . Furthermore, let f be the technology function, which assigns to each total effort $x_N = \sum_{i \in N} x_i$ the production per unit $f(x_N)$. Thus $x_N f(x_N)$ is the total production. The function f is supposed to be positive, strictly decreasing and concave on an interval $[0, \bar{x}]$, and null thereafter so that $f(0) > q$ and $f(\bar{x}) = 0$. This reflects that the more the CPR is exploited, the less it is productive.

Unlike [Funaki and Yamato \(1999\)](#), we assume that the players would like to agree upon a distribution method of the total production, prior to choosing their efforts. For this purpose, define the additive TU-game v_x so that $v_x(S) = f(x_N) \sum_{i \in S} x_i$ represents S 's total production when the overall effort in the society is given by x_N . The aforementioned distribution will be implemented by an efficient value φ . Therefore the income of player i is defined by $\theta_i(x) = \varphi_i(v_x) - qx_i$ if we standardize to 1 the price of a unit of CPR (or if cost q is measured in CPR unit). Thus, each value φ induces a non-cooperative game $(N, (\mathbb{R}^+, \theta_i)_{i \in N})$.

Let \widehat{x}_N be the total effort that achieves the social optimum, i.e. the greatest total of incomes. Indeed $\sum_{i \in N} \theta_i(x) = x_N(f(x_N) - q)$ is maximum when the following equation, independent of φ and n , is satisfied:

$$\psi(\widehat{x}_N) = q \quad (2.10)$$

where $\psi(t) = f(t) + tf'(t)$ for $t \in \mathbb{R}^+$ is a strictly decreasing function on $[0, \bar{x}]$. We also have $\widehat{x}_N \in]0, \bar{x}[$.

In this illustration, we aim at implementing the social optimum by a Nash equilibrium through the choice of a value φ . For any value φ , denote by x^φ any pure-strategy Nash equilibrium of $(N, (\mathbb{R}^+, \theta_i)_{i \in N})$ if there exists one. Let us start by two particular cases. For $\varphi = \text{ESD}$, one has $\varphi_i(v_x) = x_i f(x_N)$ so that $\theta_i(x) = x_i(f(x_N) - q)$ and the first order condition is $x_i^{\text{ESD}} f'(x_N^{\text{ESD}}) + f(x_N^{\text{ESD}}) = q$. Averaging these conditions gives:

$$\frac{x_N^{\text{ESD}} f'(x_N^{\text{ESD}})}{n} + f(x_N^{\text{ESD}}) = q$$

so that x^{ESD} exists, is unique and symmetric. Moreover $\psi(\widehat{x}_N) = q > \psi(x_N^{\text{ESD}})$.

Hence:

$$\widehat{x}_N < x_N^{\text{ESD}} \quad (2.11)$$

and we find, as in [Hardin \(1968\)](#), that the CPR is overused when each player enjoys a share of the production in proportion to her effort. Note that $x_N^{\text{ESD}} < \bar{x}$.

For $\varphi = \text{ED}$, one has $\varphi_i(v_x) = x_N f(x_N)/n$ so that $\theta_i(x) = x_N f(x_N)/n - qx_i$ and the first order condition is:

$$x_N^{\text{ED}} f'(x_N^{\text{ED}}) + f(x_N^{\text{ED}}) = nq$$

so that any x such that $x_N = x_N^{\text{ED}}$ is a Nash equilibrium. Note that if $nq > f(0)$, $x_N^{\text{ED}} = 0$. Moreover $\psi(\widehat{x}_N) = q < \psi(x_N^{\text{ED}})$. Hence:

$$\widehat{x}_N > x_N^{\text{ED}} \quad (2.12)$$

and now the CPR is underused as the equal division rule gives the players no incentive to exploit the resource.

At this point, it is quite intuitive that some convex combination of ESD and ED will allow to implement the social optimum \widehat{x}_N by a Nash equilibrium. This particular class of values has otherwise a special interest in this context: it corresponds to levy a proportional tax on individual performances, which is afterward distributed equally within the society (see [Casajus, 2015](#), for an axiomatic foundation of this approach).

Thus, let us consider $\varphi = (1 - \lambda)\text{ESD} + \lambda\text{ED}$, the first order condition becomes $(1 - \lambda)(x_N^\varphi f'(x_N^\varphi) + f(x_N^\varphi)) + \lambda(x_N^\varphi f'(x_N^\varphi) + f(x_N^\varphi))/n = q$ for player i . Summing all these conditions brings the following equation:

$$nq = (n(1 - \lambda) + \lambda)f(x_N^\varphi) + \underbrace{f'(x_N^\varphi)((1 - \lambda)x_N^\varphi + \lambda x_N^\varphi)}_{x_N^\varphi}. \quad (2.13)$$

The two inequalities (2.11, 2.12) for $\lambda = 0$ and $\lambda = 1$ respectively, and the implicit function theorem applied on equation (2.13) allow to prove existence and uniqueness of a λ^* such that $x_N^\varphi = \widehat{x}_N$. To see this, note that the partial derivative of the right member of (2.13) with respect to x_N^φ is $(n(1 - \lambda) + 1 + \lambda)f'(x_N^\varphi) + f''(x_N^\varphi)x_N^\varphi < 0$ so that $x_N^\varphi(\lambda) \in \mathcal{C}^1([0, 1])$. Note that the partial derivative of the right member

of (2.13) with respect to λ is $(1-n)f(x_N^\varphi) < 0$ so that $dx_N^\varphi/d\lambda < 0$. Substitute $nq = (n-1)q + \psi(\widehat{x}_N)$ and $x_N^\varphi = \widehat{x}_N$ in equation (2.13) finally brings the following result.

Result 2.2. *There exists a unique internal tax λ^* which allows to implement the social optimum of a CPR consumption by a unique Nash equilibrium x^φ through a redistribution $\varphi = \lambda^*ED + (1-\lambda^*)ESD$ of the total production. One has $x_i^\varphi = \widehat{x}_N/n$ for all player $i \in N$. Moreover,*

$$\lambda^* = 1 - \frac{q}{f(\widehat{x}_N)} \quad (2.14)$$

does not depend on the population's size n . Finally, the Nash equilibrium x^φ is strong.

Proof. It remains to prove that the Nash equilibrium $x^\varphi = (\widehat{x}_N/n)_{i \in N}$ is strong for the non-cooperative game $(N, (\mathbb{R}^+, \theta_i)_{i \in N})$ defined by the value $\varphi = \lambda^*ED + (1-\lambda^*)ESD$. For any coalition $S \subseteq N$, define the vector $x_{-S} = (\widehat{x}_N/n)_{i \in N \setminus S}$, the real number $\widehat{x}_{-S} = (n-s)\widehat{x}_N/n$ and, for all vector $x = (x_i)_{i \in S} \in \mathbb{R}^S$, the sum of utility functions of players in S :

$$\Theta_S(x_S) = f(x_S + \widehat{x}_{-S}) \left(s\lambda^* \frac{x_S + \widehat{x}_{-S}}{n} + (1-\lambda^*)x_S \right) - qx_S$$

where $x_S = \sum_{i \in S} x_i \in \mathbb{R}$. Let us show that $\Theta_S(x_S)$ reaches its maximum when $x_S = s \cdot \widehat{x}_N/n$:

$$\begin{aligned} \Theta'_S\left(\frac{s}{n}\widehat{x}_N\right) &= -q + f'(\widehat{x}_N) \cdot \frac{s\widehat{x}_N}{n} + f(\widehat{x}_N) \left(1 - \lambda^* + \frac{s\lambda^*}{n} \right) \\ &\stackrel{(2.10)}{=} f'(\widehat{x}_N)\widehat{x}_N \cdot \left(\frac{s}{n} - 1 \right) + \lambda^* f(\widehat{x}_N) \left(\frac{s}{n} - 1 \right) \\ &\stackrel{(2.10)}{=} \left(\frac{s}{n} - 1 \right) \cdot (q - f(\widehat{x}_N) + \lambda^* f(\widehat{x}_N)) \\ &\stackrel{(2.14)}{=} 0 \end{aligned}$$

Moreover, one may show that $\Theta''_S(x_S) < 0$ for $x_S \in [0, \bar{x}]$ so that Θ_S is a strictly concave function and has at most one maximum. ■

Lastly, let us show that $d\lambda^*/dq < 0$. Starting by differentiating $\widehat{x}_N(q)$ accord-

ingly to the implicit equation (2.10), we have:

$$\widehat{x}_N'(q) = \frac{1}{2f'(\widehat{x}_N) + \widehat{x}_N f''(\widehat{x}_N)} < 0.$$

A straight computation gives:

$$\begin{aligned} \frac{d\lambda^*}{dq} &= -\frac{f(\widehat{x}_N) - qf'(\widehat{x}_N)\widehat{x}_N'(q)}{f(\widehat{x}_N)^2} \\ &= \frac{-\widehat{x}_N'(q)}{f(\widehat{x}_N)^2} \left(f(\widehat{x}_N)(2f'(\widehat{x}_N) + \widehat{x}_N f''(\widehat{x}_N)) - qf'(\widehat{x}_N) \right) \\ &\stackrel{(2.10)}{=} \underbrace{\frac{-\widehat{x}_N'(q)}{f(\widehat{x}_N)^2}}_{>0} \left(\underbrace{f(\widehat{x}_N)f'(\widehat{x}_N)}_{<0} + \underbrace{\widehat{x}_N(f(\widehat{x}_N)f''(\widehat{x}_N) - f'(\widehat{x}_N)^2)}_{<0} \right) < 0 \end{aligned}$$

To conclude, this organization can be interpreted as a cooperative company whose owners / workers are remunerated partly by their individual efforts and partly by an equal pension levied through an internal tax. We have also shown that the harder the CPR to exploit, the lesser should be the internal tax, in order to encourage players to reach the social optimum. A large literature tackles this crucial tragedy of common-pool resources overuse. Let us emphasize that our approach internalizes the Nash implementation locally without market or social planner and is resistant to coalition formation.

2.5 Concluding remarks

An ultimate argument in favor of **NEL** is that **NEL** implies the following axiom:

Balanced cycle contributions under nullification, BCyCN.(Béal et al., 2016)

For all $v \in \mathbb{V}$, all ordering (i_1, \dots, i_n) of N ,

$$\sum_{p=1}^n \left(\varphi_{i_p}(v) - \varphi_{i_p}(v^{i_{p+1}}) \right) = \sum_{p=1}^n \left(\varphi_{i_p}(v) - \varphi_{i_p}(v^{i_{p-1}}) \right)$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Indeed, the term $\varphi_{i_{p-1}}(v) - \varphi_{i_{p-1}}(v^{i_p})$ in the left hand side of the preceding equation corresponds to the payoff variation of player i_{p-1} when player i_p is nullified,

whereas the term $\varphi_{i_{p+1}}(v) - \varphi_{i_{p+1}}(v^{i_p})$ in the right hand side corresponds to the payoff variation of player i_{p+1} when player i_p is nullified. These terms are equal if **NEL** is invoked. **BCyCN** is an interesting and very weak axiom as, following ([Béal et al., 2016](#)), any linear symmetric value satisfies it.

Bibliography

- Béal, S., Casajus, A., Hüttner, F., Rémila, E., Solal, P., 2014. Solidarity within a fixed community. *Economics Letters* 125, 440–443.
- Béal, S., Ferrières, S., Rémila, E., Solal, P., 2016. Axiomatic characterizations under players nullification. *Mathematical Social Sciences* 80, 47–57.
- Béal, S., Rémila, E., Solal, P., 2015. Axioms of invariance for TU-games. *International Journal of Game Theory* 44, 891–902.
- Casajus, A., 2015. Monotonic redistribution of performance-based allocations: A case for proportional taxation. *Theoretical Economics* 10, 887–892.
- Casajus, A., Hüttner, F., 2013. Null players, solidarity, and the egalitarian shapley values. *Journal of Mathematical Economics* 49, 58–61.
- Casajus, A., Hüttner, F., 2014. Weakly monotonic solutions for cooperative games. *Journal of Economic Theory* 154, 162–172.
- Chun, Y., Park, B., 2012. Population solidarity, population fair-ranking, and the egalitarian value. *International Journal of Game Theory* 41, 255–270.
- Funaki, Y., Yamato, T., 1999. The core of an economy with a common pool resource: A partition function form approach. *International Journal of Game Theory* 28, 157–171.
- Hardin, G., 1968. The tragedy of the Commons. *Science* 162, 1243–1248.
- Ju, Y., Borm, P., Ruys, P., 2007. The consensus value: a new solution concept for cooperative games. *Social Choice and Welfare* 28, 685–703.
- Kalai, E., 1977. Proportional solutions to bargaining situations: interpersonal utility comparisons. *Econometrica* 45, 1623–1630.

- Maschler, M., Owen, G., 1989. The consistent Shapley value for hyperplane games. *International Journal of Game Theory* 18, 389–407.
- Maschler, M., Peleg, B., 1966. A characterization, existence proof and dimension bounds for the kernel of a game. *Pacific Journal of Mathematics* 18, 289–328.
- Ostrom, E., Gardner, R., Walker, J., 1994. Rules, games, and common-pool resources. University of Michigan Press.
- Owen, G., 1972. Multilinear extensions of games. *Management Science* 18, 64–79.
- Pratt, J. W., 1964. Risk Aversion in the Small and in the Large. *Econometrica* 32, 122–136.
- Shapley, L. S., 1953. A value for n -person games. In: *Contribution to the Theory of Games vol. II* (H.W. Kuhn and A.W. Tucker eds). *Annals of Mathematics Studies* 28. Princeton University Press, Princeton.
- Thomson, W., 2011. Fair allocation rules. *Handbook of social choice and welfare* (K. Arrow, A. Sen, and K. Suzumara, eds), North-Holland 2, 393–506.
- Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. *Mathematical Social Sciences* 74, 41–59.
- van den Brink, R., 2007. Null or nullifying players: the difference between the Shapley value and equal division solutions. *Journal of Economic Theory* 136, 767–775.
- van den Brink, R., Chun, Y., Funaki, Y., Park, B., 2016. Consistency, population solidarity, and egalitarian solutions for TU-games. *Theory and Decision* (forthcoming), 1–21.
- van den Brink, R., Funaki, Y., 2009. Axiomatizations of a class of equal surplus sharing solutions for TU-games. *Theory and Decision* 67, 303–340.
- Weber, R. J., 1988. Probabilistic values for games. *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, 101–119.

Chapter 3

The proportional Shapley value and an application

Dans ce chapitre, nous définissons une valeur de Shapley pondérée mais non linéaire dont les poids correspondent aux capacités des singletons. Ils dépendent donc du jeu coopératif considéré. Comme elle partage les dividendes de Harsanyi ([Harsanyi, 1959](#)) proportionnellement à ces capacités individuelles, nous la nommons valeur de Shapley proportionnelle. Nous montrons que cette valeur recommande une allocation particulièrement intéressante dans le cadre d'une économie de production agricole, introduite par [Shapley and Shubik \(1967\)](#). Bien qu'elle ne satisfait pas les axiomes classiques de linéarité et de cohérence ([Hart and Mas-Colell, 1989](#)), nous pouvons fournir des caractérisations de notre valeur comparables à celles de la valeur de Shapley en affaiblissant ces deux axiomes. Par ailleurs, la valeur de Shapley proportionnelle hérite de plusieurs propriétés bien connues des valeurs de Shapley pondérées.

Authors: Sylvain Béal, Sylvain Ferrières, Éric Rémila, Philippe Solal

Status: Submitted

Abstract:

We introduce a non linear weighted Shapley value for cooperative games with transferable utility, in which the weights are endogenously given by the players' stand-alone worths. We call it the proportional Shapley value since it distributes the Harsanyi dividend ([Harsanyi, 1959](#)) of all coalitions in proportion to the stand-alone worths of its members. We show that this value recommends an appealing payoff distribution in a land production economy introduced in [Shapley and Shubik \(1967\)](#). Although the proportional Shapley value does not satisfy the classical axioms of linearity and consistency ([Hart and Mas-Colell, 1989](#)), the main results provide comparable axiomatic characterizations of our value and the Shapley value by means of weak versions of these two axioms. Moreover, our value inherits several well-known properties of the weighted Shapley values.

Keywords: (Weighted) Shapley value, proportionality, Harsanyi dividends, potential, land production economy.

3.1 Introduction

The Shapley value (Shapley, 1953b) is a central tool in game theory, and has received considerable attention in numerous fields and applications. Moretti and Patrone (2008) and other articles in the same issue survey several examples. Many axiomatic characterizations have helped to understand the mechanisms underlying the Shapley value, and compare it to other types of values. Shapley's original characterization (Shapley, 1953b) and the one in Shubik (1962) rely on the axiom of additivity/linearity. In Myerson (1980), the axiom of balanced contribution requires that if a player leaves a game, then the payoff variation for another player is identical to his/her own payoff variation if this other player leaves the game. Young (1985) invokes an invariance principle: a player should obtain the same payoff in two games in which all his/her contributions to coalitions are identical. Harsanyi (1959) proposes an interpretation of the Shapley value in terms of the coalitions' dividends. Roughly speaking, the Harsanyi dividend of a coalition measures the coalition's contribution to the worth of the grand coalition. The Shapley value splits equally the dividend of each coalition among its members. This interpretation has given rise to other solution concepts related to the Shapley value such as the selectope (Hammer et al., 1977) and the weighted Shapley values, originally introduced in Shapley (1953a) but popularized later by Kalai and Samet (1987). The selectope is the convex hull of the payoff vectors obtained by assigning the Harsanyi dividends to the associated coalitions' members. A weighted Shapley value splits the Harsanyi dividends in proportional to the exogenously given weights of its members. Both solution concepts are linear. The Harsanyi dividends are also often employed to compare different values (see section 5 in Herings et al., 2008; van den Brink et al., 2011, for instance).

In this article, we introduce a value based on another distribution of the Harsanyi dividends. It is similar in spirit to the weighted Shapley values, except that the weights are endogenous: they are given by the stand-alone worths of the players. Thus it coincides with the Shapley value whenever all such worths are equal. We call our value the proportional (weighted) Shapley value. The proportional principle incorporated to this value is often considered as intuitive in various classes of sharing problems (see Moulin, 1987, for instance).¹ Although the proportional

¹Other values incorporating some degree of proportionality are the proportional value (Ort-

Shapley value is non linear, it admits a close form and operational expression. It also satisfies many classical axioms such as efficiency and the dummy player property, and preserves the equal treatment property contrary to the asymmetric weighted Shapley values. The proportional Shapley value is well-defined for games in which the worths of all singleton coalitions have the same sign. This (not so) restrictive class of games includes several applications, such as airport games ([Littlechild and Owen, 1973](#)), auction games ([Graham et al., 1990](#)), carpool problems ([Naor, 2005](#)) and data sharing games ([Dehez and Tellone, 2013](#)). In airport games, a player is characterized by a positive real number (his/her “cost”), and the worth of the associated singleton coalition is equal to this number. So, it makes sense to use these numbers to define weights. In this article, we focus on the land production economies introduced by [Shapley and Shubik \(1967\)](#) in order to underline that the proportional Shapley value prescribes particularly relevant payoff distributions, especially compared to the (weighted) Shapley value(s). An expression of the Shapley value for land production economies is also given.

The rest of our contributions can be described as follows.

Firstly, the proportional Shapley value inherits some of the results concerning the weighted Shapley values. In particular, we can easily adapt the characterization in [Myerson \(1980\)](#) by using an axiom of proportional balanced contributions, and the characterization in [Hart and Mas-Colell \(1989\)](#) by constructing a proportional potential function. This part also includes a recursive formula inspired by the recursive formula of the Shapley value in [Maschler and Owen \(1989\)](#) and underlines that the proportional Shapley value of any convex game is in the core as a corollary of a result in [Monderer et al. \(1992\)](#). The proofs of these benchmark results are straightforward and omitted. In fact, any result stated for the weighted Shapley values on a class of games built from a fixed characteristic function and its subgames also holds for the proportional Shapley value. A similar result is pointed out by [Neyman \(1989\)](#), who shows that [Shubik \(1962\)](#)’s axiomatic characterization of the Shapley value still holds if the axioms are applied to the additive group generated by the considered game and the games obtained from it after the nullification of any coalition (called subgames by Neyman).

Secondly, as soon as we consider a class of games with varying characteristic functions, the immediate transposition of existing results is no longer possible. For [mann, 2000](#)) and the proper Shapley values ([van den Brink et al., 2015](#)).

instance, the proportional Shapley value does not satisfy the classical axioms of linearity and consistency (Hart and Mas-Colell, 1989). Nevertheless, weak versions of these two axioms can be invoked (and even combined) to provide comparable axiomatic characterizations of the proportional Shapley value and the Shapley value, in the sense that these results only differ with respect to one axiom. Both characterizations have in common the well-known axioms of efficiency and dummy player out (Derks and Haller, 1999), which states that the payoff of a player is not affected if a dummy player leaves the game, and our weak version of linearity. More specifically, Proposition 3.5 shows that if two values satisfy efficiency, dummy player out and weak linearity, and if they coincide on games that are additive except, possibly, for the grand coalition, then they must be equal. In other words, there exists a unique extension of a value defined on these almost additive games to the set of all games in the much larger class we consider. The proof of this result emphasizes that tools from linear algebra can still be used on a class of games that is not a vector space.

Thirdly, Proposition 3.7 invokes the weak version of the axiom of consistency in addition to the three axioms appearing in the previous result. It turns out that a value satisfying these four axioms is the Shapley value if and only if it also satisfies the classical axiom of standardness, and is the proportional Shapley value if and only if it also satisfies a natural proportional version of standardness. The later axioms requires, in two-player games, that each player obtains first his/her stand alone worth plus a share of what remains of the worth of the grand coalition that is proportional to his/her stand-alone worth. It is worth noting that the two values are distinguished by axioms on two-player games only. Similarly, in addition to the three axioms appearing in Proposition 3.5, Proposition 3.8 invokes two new axioms inspired by the axiom of aggregate monotonicity in Megiddo (1974). More specifically, these axioms examine the consequences of a change in the worth of the grand coalition, *ceteris paribus*. Equal aggregate monotonicity requires equal payoff variations for all players, while proportional aggregate monotonicity requires payoff variations in proportion to the players' stand-alone worths. Among the values satisfying efficiency, dummy player out and weak linearity, the Shapley value is the only one that also satisfies equal aggregate monotonicity, and the proportional Shapley value is the only one that also satisfies proportional aggregate monotonicity.

Fourthly, the results presented so far all involve variable player sets, since they invoke axioms such as dummy player out and consistency. It is however possible

to characterize the proportional Shapley value on a class of games with a fixed player set. In order to do so, we introduce another variant of the axiom of balanced contributions in which the removal of a player is replaced by his/her dummification. A player's dummification refers to his/her complete loss of synergy, in the sense that the worth of any coalition containing this player is now identical to that of the same coalition without this player plus his/her stand alone worth. In other words, the player becomes dummy, while the worth of any coalition not containing him/her remains unchanged. The dummification is in essence similar to the nullification of a player studied by [Gómez-Rúa and Vidal-Puga \(2010\)](#), [Béal et al. \(2014\)](#) and [Béal et al. \(2016\)](#). The new axiom of balanced contributions under dummification requires, for any two players, equal allocation variation after the dummification of the other player. Combined with efficiency and the classical axiom of inessential game property (each player obtains his/her stand-alone worth in case the game is additive), this axiom characterizes the proportional Shapley value value.

The rest of the article is organized as follows. Section [3.2](#) provides definitions and introduces the proportional Shapley value. It also states a first result for the case of land production economies. Section [3.3](#) briefly states properties of the proportional Shapley value that are inherited from the literature on the weighted Shapley values. Section [3.4](#) contains the main axiomatic characterizations, relying on the weak version of linearity, consistency, and on balanced contributions under dummification. Section [4.5](#) concludes. The appendix contains the results on the land production economies that are not stated in section [3.2](#), some technical proofs and the proofs of logical independence of the axioms used in some results.

3.2 Definitions, notation and motivation

3.2.1 Notation

We denote by \mathbb{R} , \mathbb{R}_+ , \mathbb{R}_{++} and \mathbb{R}^* the sets of all real numbers, non-negative real numbers, positive real numbers and non-null real numbers respectively. For a real number $b \in \mathbb{R}$ we shall also use notation $|b|$ for the absolute value of b . In order to denote the cardinality of any finite set S , the same notation $|S|$ will sometimes be used without any risk of confusion, but we shall often write s for simplicity.

3.2.2 Cooperative games with transferable utility

Let $\mathcal{U} \subseteq \mathbb{N}$ be a fixed and infinite universe of players. Denote by U the set of all finite subsets of \mathcal{U} . A **cooperative game with transferable utility**, or simply a **game**, is a pair (N, v) where $N \in U$ and $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For a game (N, v) , we write (S, v) for the **subgame** of (N, v) induced by $S \subseteq N$ by restricting v to 2^S . For $N \in U$ and $a \in \mathbb{R}^N$, denote by (N, v_a) the **additive game on N induced by a** , i.e. $v_a(S) = \sum_{j \in S} a_j$ for all $S \in 2^N$.

Define \mathcal{C} as the class of all games with a finite player set in U and \mathcal{C}_N as the subclass of \mathcal{C} containing the games with player set N . A game (N, v) is **individually positive** if $v(\{i\}) > 0$ for all $i \in N$ and **individually negative** if $v(\{i\}) < 0$ for all $i \in N$. Let \mathcal{C}^0 denote the class containing all individually positive and individually negative games, and \mathcal{C}_N^0 the intersection of \mathcal{C}^0 and \mathcal{C}_N . For $N \in U$ and $a \in \mathbb{R}_{++}^N$, define the subclass of \mathcal{C}_N^a containing all games such that the singleton worths are obtained by multiplying vector a by some non-null real number, that is:

$$\mathcal{C}_N^a = \{(N, v) \in \mathcal{C}_N \mid \exists c \in \mathbb{R}^* : \forall i \in N, v(\{i\}) = ca_i\}.$$

Thus, if $a' \in \mathbb{R}_{++}^N$ is multiple of $a \in \mathbb{R}_{++}^N$, then $\mathcal{C}_N^a = \mathcal{C}_N^{a'}$, and $\mathcal{C}_N^0 = \bigcup_{a \in \mathbb{R}_{++}^N} \mathcal{C}_N^a$. Finally, let \mathcal{A}^0 and \mathcal{A}_N^0 denote the subclasses of additive games in \mathcal{C}^0 and \mathcal{C}_N^0 respectively.

For all $b \in \mathbb{R}$, all $(N, v), (N, w) \in \mathcal{C}$, the game $(N, bv + w) \in \mathcal{C}$ is defined as $(bv + w)(S) = bv(S) + w(S)$ for all $S \in 2^N$. The **unanimity game** on N induced by a nonempty coalition S , denoted by (N, u_S) , is defined as $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. Since [Shapley \(1953b\)](#), it is well-known that each function v admits a unique decomposition into unanimity games:

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_v(S) u_S$$

where $\Delta_v(S)$ is the **Harsanyi dividend** ([Harsanyi, 1959](#)) of S , defined as $\Delta_v(S) = v(S) - \sum_{T \in 2^S \setminus \{\emptyset\}} \Delta_v(T)$. The Harsanyi dividend of S represents what remains of $v(S)$ once the dividends of all nonempty subcoalitions of S have been distributed. A player $i \in N$ is **dummy** in (N, v) if $v(S) - v(S \setminus \{i\}) = v(\{i\})$ for all $S \in 2^N$ such that $S \ni i$. Let $D(N, v)$ be the set of dummy players in (N, v) . Two distinct players $i, j \in N$ are **equal** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{N \setminus \{i, j\}}$.

3.2.3 Values

A **value** on \mathcal{C} (respectively on \mathcal{C}^0) is a function f that assigns a payoff vector $f(N, v) \in \mathbb{R}^N$ to any $(N, v) \in \mathcal{C}$ (respectively any $(N, v) \in \mathcal{C}^0$). In this article, we call upon values that admit intuitive formulations in terms of the distribution of the Harsanyi dividends.

The **Shapley value** (Shapley, 1953b) is the value Sh on \mathcal{C} defined as:

$$Sh_i(N, v) = \sum_{S \in 2^N: S \ni i} \frac{1}{s} \Delta_v(S), \quad \forall (N, v) \in \mathcal{C}, \forall i \in N.$$

For each $i \in \mathcal{U}$ let $w_i \in \mathbb{R}_{++}$, and $w = (w_i)_{i \in \mathcal{U}}$. The **(positively) weighted Shapley value** (Shapley, 1953b) with weights w is the value Sh^w on \mathcal{C} defined as:²

$$Sh_i^w(N, v) = \sum_{S \in 2^N: S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S), \quad \forall (N, v) \in \mathcal{C}, \forall i \in N.$$

The **proportional Shapley value** is the value PSh on \mathcal{C}^0 defined as:

$$PSh_i(N, v) = \sum_{S \in 2^N: S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S), \quad \forall (N, v) \in \mathcal{C}^0, \forall i \in N.$$

Thus, the Harsanyi dividend of a coalition S is shared equally among its members in the Shapley value, in proportion to exogenous weights in a positively weighted Shapley value, and in proportion to the stand-alone worths of its members in the PSh value. As a consequence, the Shapley and PSh values coincide whenever all singleton worths are equal.

3.2.4 A motivating example: Land production economies

Consider a set $N = \{1, \dots, n\}$ of peasants and an amount of land $L \in \mathbb{R}_{++}$. Shapley and Shubik (1967, section VI) model the production process of several laborers working together by a function $\phi : N \times \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ which specifies the output

$$\phi(s, l) = \frac{l}{L} z(s) \tag{3.1}$$

²Weighted Shapley values with possibly null weights are defined in Shapley (1953a), and studied in Kalai and Samet (1987) and Monderer et al. (1992), among others.

achieved by s identical laborers from an area of land $l \leq L$, where $z(s) := \phi(s, L)$, and $z(s) > 0$ whenever $s > 0$.³ They assume that each farmer owns the same share of L , which leads to an associated game by setting $v(S) = \phi(s, sL/n) = sz(s)/n$ for all $S \in 2^N$. A consequence of the symmetry in this model is that both PSH and Sh yield an equal split of the total output.

As suggested by [Shapley and Shubik \(1967\)](#), it makes sense to introduce some heterogeneity by considering that each peasant owns an amount of land $a_i \in \mathbb{R}_{++}$, such that $\sum_{i \in N} a_i = L$. Let $a := (a_i)_{i \in N}$. Since the output only depends on function z , a **land production economy** can be described by a triple (N, a, z) . For any land production economy (N, a, z) , the associated game $(N, v_{a,z})$ assigns to each coalition S a worth

$$v_{a,z}(S) = \phi\left(s, \sum_{i \in S} a_i\right) = \frac{\sum_{i \in S} a_i}{L} z(s)$$

for any coalition of farmers S . Note that $(N, v_{a,z}) \in \mathcal{C}_N^a$. In this asymmetric version of the model, [Shapley and Shubik \(1967\)](#) do not provide a formulation of the Shapley value, which is not easy to compute. In the appendix, we provide a close form expression of the Shapley value, which is nonetheless much less interpretable and appealing than the expression of the proportional Shapley value below. Proposition 3.1 shows that PSH can be considered as a relevant alternative to the Shapley value in the asymmetric land production economy. The proof is also relegated to the appendix.

Proposition 3.1. *For any land production economy (N, a, z) and any $i \in N$, it holds that*

$$PSH_i(N, v_{a,z}) = \frac{a_i}{L} z(n).$$

The meaning of Proposition 3.1 is clear. The output $z(n)$ produced by the n farmers altogether is shared in proportion to the landholdings. Proposition 3.1 also emphasizes situations in which the proportional Shapley value can be more suitable than the weighted Shapley values. After all, it is true that $PSH(N, v_{a,z})$ in Proposition 3.1 can be obtained as a weighted Shapley value by choosing the landholdings $(a_i)_{i \in N}$ as weights. But now, suppose that a farmer buys a part, but not all, of the landholding of another farmer, *ceteris paribus*. The new production

³Rather than ϕ , [Shapley and Shubik \(1967\)](#) use function ϕ^* defined as $\phi^*(s, l) = \max_{t \in \{1, \dots, s\}} \phi(t, l)$ for all pairs $(s, l) \in N \times \mathbb{R}_{++}$. The result in this section and in the appendix holds if ϕ is replaced by ϕ^* .

economy is characterized by the same player set. Because the weights in a weighted Shapley value do not change with the characteristic function, they must remain the same in the new land production economy. As a consequence, the original weighted Shapley value applied to this new problem is likely to be less suitable. To the contrary, if the proportional Shapley value is applied to both situations, the weights associated to the players adjust accordingly to account for the new landholdings.

3.3 Legacy results

All the results in this section are based on the following useful property of PSh .

Lemma 3.1. *For each game $(N, v) \in \mathcal{C}^0$, define the weights $w(v) = (w_i(v))_{i \in N}$ such that $w_i(v) = |v(\{i\})|$ for all $i \in N$. Then, it holds that $PSh(N, v) = Sh^{w(v)}(N, v)$.*

The proof is obvious and omitted. Lemma 3.1 does not mean that PSh is a weighted Shapley value since the weights $w(v)$ can be different for two games defined on the same player set. Nevertheless, Lemma 3.1 is sufficient to adapt well-known results in the literature that involve a game with a fixed characteristic function and its subgames. As a first example, we consider the characterizations obtained by Myerson (1980) and Hart and Mas-Colell (1989) by means of the next axioms.

Balanced contributions (BC). For all $(N, v) \in \mathcal{C}$, all $i, j \in N$,

$$f_i(N, v) - f_i(N \setminus \{j\}, v) = f_j(N, v) - f_j(N \setminus \{i\}, v).$$

w -balanced contributions (w -BC). For all $w = (w_i)_{i \in U}$ with $w_i \in \mathbb{R}_{++}$ for all $i \in U$, all $(N, v) \in \mathcal{C}$, all $i, j \in N$,

$$\frac{f_i(N, v) - f_i(N \setminus \{j\}, v)}{w_i} = \frac{f_j(N, v) - f_j(N \setminus \{i\}, v)}{w_j}.$$

Myerson (1980) characterizes the Shapley value by **BC** and **E**.

Efficiency (E). $\sum_{i \in N} f_i(N, v) = v(N)$.

Hart and Mas-Colell (1989) demonstrate that the class of weighted Shapley values coincides with the values satisfying **E** and w -**BC** for all possible weights w . A natural variant of w -**BC** requires, for any two players, an allocation variation for each of them after the other player has left that is proportional to their stand-alone worth.

Proportional balanced contributions (PBC). For all $(N, v) \in \mathcal{C}^0$, all $i, j \in N$,

$$\frac{f_i(N, v) - f_i(N \setminus \{j\}, v)}{v(\{i\})} = \frac{f_j(N, v) - f_j(N \setminus \{i\}, v)}{v(\{j\})}.$$

The main notable difference between **PBC** and w -**BC** is that our weights are endogenous, i.e. they can vary across games. The consequence is that the system of equations generated by **PBC** together with **E** is not linear. Nevertheless, it gives rise to a unique non-linear value.

Proposition 3.2. *The proportional Shapley value is the unique value on \mathcal{C}^0 that satisfies **E** and **PBC**.*

Using **PBC** and the fact that PSH satisfies **E**, it is possible to obtain a recursive formula of PSH very close to the recursive formula of the Shapley value provided by Maschler and Owen (1989). For all $(N, v) \in \mathcal{C}^0$ and all $i \in N$,

$$PSH_i(N, v) = \sum_{j \in N \setminus \{i\}} \frac{v(\{j\})}{\sum_{k \in N} v(\{k\})} PSH_i(N \setminus \{j\}, v) + \frac{v(\{i\})}{\sum_{k \in N} v(\{k\})} (v(N) - v(N \setminus \{i\})).$$

The latter expression is similar to following recursive formula for the Shapley value (Maschler and Owen, 1989):

$$Sh_i(N, v) = \sum_{j \in N \setminus \{i\}} \frac{1}{n} Sh_i(N \setminus \{j\}, v) + \frac{1}{n} (v(N) - v(N \setminus \{i\})).$$

Connected to Myerson's approach is the fundamental notion of potential introduced by Hart and Mas-Colell (1989). For any system of weights w , the unique w -potential function P_w is defined as $P_w(\emptyset, v) = 0$ and as $\sum_{i \in N} w_i (P_w(N, v) - P_w(N \setminus \{i\}, v)) = v(N)$. They show that the weighted Shapley value with weights w in game (N, v) coincides with payoffs $w_i (P_w(N, v) - P_w(N \setminus \{i\}, v))$, $i \in N$. Below is an adaptation for PSH . A **proportional potential function** is a function $Q : \mathcal{C}^0 \rightarrow \mathbb{R}$ such that

$Q(\emptyset, v) = 0$ and for all $(N, v) \in \mathcal{C}^0$,

$$\sum_{i \in N} |v(\{i\})| (Q(N, v) - Q(N \setminus \{i\}, v)) = v(N). \quad (3.2)$$

The following proposition mimics Theorem 5.2 in [Hart and Mas-Colell \(1989\)](#).

Proposition 3.3. *There exists a unique proportional potential function Q on \mathcal{C}^0 . Moreover, for each game $(N, v) \in \mathcal{C}^0$, it holds that $Q(N, v) = P_{w(v)}(N, v)$. Thus, for each game $(N, v) \in \mathcal{C}^0$ and each $i \in N$,*

$$PSh_i(N, v) = |v(\{i\})| (Q(N, v) - Q(N \setminus \{i\}, v)).$$

Finally Q can be computed recursively by the following formula:

$$Q(N, v) = \frac{1}{\sum_{i \in N} |v(\{i\})|} \left(v(N) + \sum_{j \in N} |v(\{j\})| Q(N \setminus \{j\}, v) \right).$$

It suffices to define Q on \mathcal{C}^0 as $Q(N, v) = P_{w(v)}(N, v)$ for all $(N, v) \in \mathcal{C}^0$. In a sense, Q is a normalized (or dimensionless) potential because for each $i \in N$, $Q(\{i\}, v) = 1$ if $v(\{i\}) > 0$ and $Q(\{i\}, v) = -1$ if $v(\{i\}) < 0$. Dimensionless numbers are often desirable, in particular in economics (elasticities). The proportional potential will play a key role in some of the main results in the next section. We conclude this section by stating a sufficient condition under which the PSh value lies in the core. The core of a game $(N, v) \in \mathcal{C}^0$ is the (possibly empty) set $C(N, v) = \{x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ and } \sum_{i \in N} x_i = v(N)\}$. [Shapley \(1971\)](#) shows that the Shapley value belongs to the core of a convex game. [Monderer et al. \(1992\)](#) generalize this result and prove that the core of a convex game coincides with the set of weighted Shapley values (with possibly null weights). Building on this result, it is immediate to prove the PSh value lies in the core of a convex game.

Proposition 3.4. *If $(N, v) \in \mathcal{C}^0$ is convex, then $PSh(N, v) \in C(N, v)$.*

Not surprisingly, PSh is not necessarily a core imputation in a non-convex

game with a nonempty core as shown in the following example.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	4	4	1	12	-5	15	22
$\Delta_v(S)$	4	4	1	4	-10	10	9

The core of (N, v) is not empty since it contains, for instance, the Shapley value $Sh(N, v) = (4, 14, 4)$. However, $PSh(N, v) = (2, 18, 2)$ is not in the core.

3.4 Main results

The results in this section rely on weak versions of the axioms of linearity and consistency as proposed by [Hart and Mas-Colell \(1989\)](#), and on another variant of PBC.

3.4.1 Potential, linearity and consistency

Contrary to the potential approach, the well-known axioms of linearity and consistency require less evident modifications in order to account for PSh , even if these axioms are satisfied by any weighted Shapley value. We examine each case separately before combining them in order to characterize PSh .

Linearity (L). For all $b \in \mathbb{R}$, all $(N, v), (N, w) \in \mathcal{C}$, $f(N, bv + w) = bf(N, v) + f(N, w)$.

The class \mathcal{C}^0 is not a vector space. Even if it is required that the game $(N, bv + w)$ constructed in the previous definition still belongs to \mathcal{C}^0 , it is clear that PSh violates this adaptation of **L** on \mathcal{C}^0 . Nonetheless, PSh satisfies the following weaker version of the axiom.

Weak linearity (WL). For all $a \in \mathbb{R}_{++}^N$, all $b \in \mathbb{R}$, all $(N, v), (N, w) \in \mathcal{C}_N^a$, if $(N, bv + w) \in \mathcal{C}_N^a$ then $f(N, bv + w) = bf(N, v) + f(N, w)$.

So **WL** only applies to games belonging to the same subclass \mathcal{C}_N^a of \mathcal{C}_N^0 , i.e.

pair of games for which the ratio $w(\{i\})/v(\{i\})$ is the same for all players. The requirement that $(N, bv + w) \in \mathcal{C}_N^a$ is necessary: in case b is equal to the opposite of the above-mentioned ration $(N, bv + w)$ would not belong to \mathcal{C}_N^a (and neither to \mathcal{C}_N^0). Before showing that PSh satisfies **WL**, we present a key result in which **WL** is combined with the classical axioms **E** and dummy player out.

Dummy player out (DPO). For all $(N, v) \in \mathcal{C}^0$, if $i \in N$ is a dummy player in (N, v) , then for all $j \in N \setminus \{i\}$, $f_j(N, v) = f_j(N \setminus \{i\}, v)$.

DPO was suggested first in [Tijs and Driessen \(1986, section V\)](#) and is closely related to the widely-used null player out axiom ([Derks and Haller, 1999](#)). Proposition 3.5 below requires the following definition. A game (N, v) is **quasi-additive** if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \in 2^N \setminus \{N\}$. Let \mathcal{QA}^0 denote the class of all quasi-additive games in \mathcal{C}^0 . In a quasi-additive game, the worths of all coalitions are additive except, possibly, for the grand coalition for which there can be some surplus or loss compared to the sum of the stand-alone worths of its members. So \mathcal{QA}^0 includes the class \mathcal{A}^0 . Proposition 3.5 essentially states that a value satisfies **E**, **DPO** and **WL** is completely determined by what it prescribes on quasi-additive games.

Proposition 3.5. *Consider two values f and g satisfying **E**, **DPO** and **WL** on \mathcal{C}^0 such that $f = g$ on \mathcal{QA}^0 . Then $f = g$ on \mathcal{C}^0 .*

The non-trivial and lengthy proof of Proposition 3.5 relies on two Lemmas. In order to lighten the exposition, this material is relegated to the appendix. A similar result can be stated on the larger class \mathcal{C} by replacing **WL** by **L**. The only change would be to consider all quasi-additive games in \mathcal{C} and not just those in \mathcal{C}^0 . At this point, remark also that both the Shapley value and PSh satisfy the three axioms invoked in Proposition 3.5.⁴ In order to compare and distinguish the two values, we present extra axioms below. The next axiom relies on the reduced game proposed by [Hart and Mas-Colell \(1989\)](#). Let f be a value on \mathcal{C} , $(N, v) \in \mathcal{C}$ and $S \in 2^N \setminus \{\emptyset\}$. The **reduced game** (S, v_S^f) induced by S and f is defined, for all $T \in 2^S$, by:

$$v_S^f(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} f_i(T \cup (N \setminus S), v), \quad (3.3)$$

⁴Formal proofs are given later on.

and resumes to $v_S^f(T) = \sum_{i \in T} f_i(T \cup (N \setminus S), v)$ if f satisfies **E**.

Consistency (C). For all $(N, v) \in \mathcal{C}$, all $S \in 2^N$, and all $i \in S$, $f_i(N, v) = f_i(S, v_S^f)$.

The Shapley value satisfies **C** on \mathcal{C} . If **C** is enunciated on \mathcal{C}^0 , the extra condition that the considered reduced game (S, v_S^f) remains in \mathcal{C}^0 must be added. Such a condition is, however, not sufficient for our objective: PSh fails to satisfy the axiom as illustrated by the following example.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	1	2	3	9	4	5	18
$\Delta_v(S)$	1	2	3	6	0	0	6

Consider player 1. It is easy to check that $PSh_1(\{1, 2, 3\}, v) = 4$. Now consider the reduced game $(\{1, 3\}, v_{\{1, 3\}}^{PSh})$, where $v_{\{1, 3\}}^{PSh}(\{1\}) = PSh_1(\{1, 2\}, v) = 3$, $v_{\{1, 3\}}^{PSh}(\{3\}) = PSh_3(\{2, 3\}, v) = 3$, and $v_{\{1, 3\}}^{PSh}(\{1, 3\}) = PSh_1(\{1, 2, 3\}, v) + PSh_3(\{1, 2, 3\}, v) = 4 + 6 = 10$. Note that $(\{1, 3\}, v_{\{1, 3\}}^{PSh}) \in \mathcal{C}^0$ and is symmetric, which implies that $PSh_1(\{1, 3\}, v_{\{1, 3\}}^{PSh}) = 5 \neq PSh_1(\{1, 2, 3\}, v) = 4$, proving that PSh does not satisfy **C** in this context.

It is possible to weaken **C** by imposing consistency of the value on the particular subclass of quasi-additive games \mathcal{QA}^0 . To this end, we begin by a Lemma stating that \mathcal{QA}^0 is almost close under the reduction operation for values satisfying the following mild condition.

Inessential game property (IGP). For all $(N, v) \in \mathcal{A}^0$, all $i \in N$, $f_i(N, v) = v(\{i\})$.⁵

Lemma 3.2. Consider any value f that satisfies **IGP** on \mathcal{QA}^0 . Then, for each $(N, v) \in \mathcal{QA}^0$ and each $S \in 2^N$ such that $s \geq 2$, it holds that $(S, v_S^f) \in \mathcal{QA}^0$. Furthermore, for each $T \in 2^S \setminus \{S\}$, $v_S^f(T) = \sum_{i \in T} v(\{i\})$.

Proof. Consider any value f that satisfies **IGP** on \mathcal{QA}^0 , any $(N, v) \in \mathcal{QA}^0$ and any $S \in 2^N$ such that $s \geq 2$. If $S = N$, the result is trivial. So let $S \neq N$, and

⁵This axiom is also called the projection axiom in [Aumann and Shapley \(1974\)](#).

consider any coalition $T \in 2^S \setminus \{S\}$. To show $v_S^f(T) = \sum_{i \in T} v_S^f(\{i\})$. By definition (3.3) of the reduced game (S, v_S^f) , the worth $v_S^f(T)$ only relies on the subgame $(T \cup (N \setminus \{S\}), v)$ of (N, v) . Since $T \neq S$, $(T \cup (N \setminus \{S\}), v)$ is a strict subgame of (N, v) , and since $(N, v) \in \mathcal{QA}^0$, we get that $(T \cup (N \setminus \{S\}), v) \in \mathcal{A}^0$. By **IGP**, f satisfies **E** in $(T \cup (N \setminus \{S\}), v)$ and $f_i(T \cup (N \setminus \{S\}), v) = v(\{i\})$ for each $i \in T \cup (N \setminus \{S\})$. This implies that $v_S^f(T) = \sum_{i \in T} f_i(T \cup (N \setminus \{S\}), v) = \sum_{i \in T} v(\{i\})$. The proof is complete since T was an arbitrary coalition in $2^S \setminus \{S\}$. ■

Remark 3.1. Lemma 3.2 excludes coalitions of size 1, i.e. reduced games with a unique player. Such reduced games may not belong to \mathcal{QA}^0 as suggested by the following generic example. For any game $(N, v) \in \mathcal{QA}^0$ such that $v(N) = 0$, we get $PSh_i(N, v) = 0$ for all $i \in N$. Therefore, for each $i \in N$, the reduced game $(\{i\}, v_{\{i\}}^{PSh})$ is such that $v_{\{i\}}^{PSh}(\{i\}) = 0$ and does not belong to \mathcal{QA}^0 . □

Next, we introduce our weak version of **C**.

Weak consistency (WC). For all $(N, v) \in \mathcal{QA}^0$, all $S \in 2^N$ such that $(S, v_S^f) \in \mathcal{QA}^0$, and all $i \in S$, $f_i(N, v) = f_i(S, v_S^f)$.

Finally, we invoke the following axioms.

Proportional standardness (PS). For all $(\{i, j\}, v) \in \mathcal{C}^0$, we have $f_i(\{i, j\}, v) = \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} v(\{i, j\})$.

This axiom is called proportionality for two person games in [Ortmann \(2000\)](#) and two-player games proportionality in [Huettner \(2015\)](#). It can be considered as the proportional counterpart of the classical axiom of standardness ([Hart and Mas-Colell, 1989](#)).

Standardness (S). For all $(\{i, j\}, v) \in \mathcal{C}^0$, $f_i(\{i, j\}, v) = v(\{i\}) + \frac{1}{2}(v(\{i, j\}) - v(\{i\}) - v(\{j\}))$.

Both axioms first assign their stand-alone worths to the two players. Then, proportional standardness splits the remaining surplus in proportion to these stand-

alone worths, while standardness shares the surplus equally, if any. To understand this interpretation, note that

$$\frac{v(\{i\})}{v(\{i\}) + v(\{j\})} v(\{i, j\}) = v(\{i\}) + \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} (v(\{i, j\}) - v(\{i\}) - v(\{j\})).$$

Proportional aggregate monotonicity (PAM). For all $b \in \mathbb{R}$, all $(N, v) \in \mathcal{C}^0$ such that $n \geq 2$, and all $i, j \in N$,

$$\frac{f_i(N, v) - f_i(N, v + bu_N)}{v(\{i\})} = \frac{f_j(N, v) - f_j(N, v + bu_N)}{v(\{j\})}.$$

The axiom compares two games that only differ with respect to the worth of the grand coalition. It states that the players enjoy payoff variations that are proportional to their stand-alone worths. Note that **PAM** is well-defined since $(N, v) \in \mathcal{C}^0$ implies that $(N, v + bu_N) \in \mathcal{C}^0$ for all $b \in \mathbb{R}$. Without the further requirement of **E**, **PAM** is not related to Aggregate monotonicity (Megiddo, 1974), which requires that none of the players should be hurt if the worth of the grand coalition increases. In fact, the Shapley value satisfies Aggregate monotonicity but not **PAM**, while the value f on \mathcal{C}^0 , which assigns to each game $(N, v) \in \mathcal{C}^0$ and to each $i \in N$, the payoff $f_i(N, v) = -v(\{i\}) / \sum_{j \in N} v(\{j\}) \times v(N)$ satisfies **PAM** but not Aggregate monotonicity. However, if a value satisfies **PAM** and **E** on \mathcal{C}^0 , then it also satisfies Aggregate monotonicity on \mathcal{C}^0 .

The next result lists which of these axioms are satisfied by PSh .

Proposition 3.6. *PSh satisfies **E**, **DPO**, **WL**, **PAM**, **WC** and **PS** on \mathcal{C}^0 .*

Proof. The proof follows from Proposition 3.2 for **E**.

Regarding **DPO**, observe that if a player $i \in N$ is dummy in (N, v) , then $\Delta_v(S) = 0$ for all $S \in 2^N$ such that $S \ni i$ and $s \geq 2$. So, for any $j \in N \setminus \{i\}$, it holds that

$$\begin{aligned} PSh_j(N, v) &= \sum_{S \in 2^N : S \ni j} \frac{v(\{j\})}{\sum_{k \in S} v(\{k\})} \Delta_v(S) \\ &= \sum_{S \in 2^{N \setminus \{i\}} : S \ni j} \frac{v(\{j\})}{\sum_{k \in S} v(\{k\})} \Delta_v(S) \\ &= PSh_j(N \setminus \{i\}, v) \end{aligned}$$

as desired.

Regarding **WL**, consider any two games $(N, v), (N, w) \in \mathcal{C}_N^a$, which means

that, for all $i \in N$, $w(\{i\}) = cv(\{i\})$ for some $c \in \mathbb{R}^*$. Note that for all nonempty $S \in 2^N$, this implies

$$\frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} = \frac{w(\{i\})}{\sum_{j \in S} w(\{j\})}. \quad (3.4)$$

Choose any $b \in \mathbb{R}$ such that $(N, bv + w) \in \mathcal{C}_N^a$ in order to compute $PSh_i(N, bv + w)$. The claim is trivial for $b = 0$. So suppose $b \in \mathbb{R}^*$. By linearity of function Δ (third equality) and equation (3.4) (fourth equality), we have

$$\begin{aligned} PSh_i(N, bv + w) &= \sum_{S \in 2^N: S \ni i} \frac{(bv + w)(\{i\})}{\sum_{j \in S} (bv + w)(\{j\})} \Delta_{bv+w}(S) \\ &= \sum_{S \in 2^N: S \ni i} \frac{(b+c)v(\{i\})}{\sum_{j \in S} (b+c)v(\{j\})} \Delta_{bv+w}(S) \\ &= \sum_{S \in 2^N: S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \left(b\Delta_v(S) + \Delta_w(S) \right) \\ &= b \sum_{S \in 2^N: S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \sum_{S \in 2^N: S \ni i} \frac{w(\{i\})}{\sum_{j \in S} w(\{j\})} \Delta_w(S) \\ &= bPSh_i(N, v) + PSh_i(N, w). \end{aligned}$$

Regarding **PAM**, pick any game $(N, v) \in \mathcal{C}^0$ and any $b \in \mathbb{R}$. Note that $\Delta_{v+bu_N}(S) = \Delta_v(S)$ for all $S \in 2^N \setminus \{N\}$, and that $\Delta_{v+bu_N}(N) = \Delta_v(N) + b$. As a consequence,

$$\frac{PSh_i(N, v) - PSh_i(N, v + bu_N)}{v(\{i\})} = \frac{-b}{\sum_{j \in N} v(\{j\})}$$

does not depend on $i \in N$, which proves that PSh satisfies **PAM**.

Regarding **WC**, consider any game $(N, v) \in \mathcal{QA}^0$, any nonempty $S \in 2^N$ and any $i \in S$. The assertion that $PSh_i(N, v) = PSh_i(S, v_S^{PSh})$ is trivial if $s = n$. Since $(N, v) \in \mathcal{QA}^0$, note that $\Delta_v(T) = 0$ for all T such that $t \in \{2, \dots, n-1\}$. As a consequence, PSh admits the following simple formulation:

$$PSh_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N). \quad (3.5)$$

The assertion is thus also obvious for $s = 1$ in case $(S, v_S^{PSh}) \in \mathcal{QA}^0$ (see Lemma 3.2). Now, let us assume that $s \in \{2, \dots, n-1\}$. By Lemma 3.2, we know that $(S, v_S^{PSh}) \in \mathcal{QA}^0$. Furthermore, as noted in Lemma 3.2, $v_S^{PSh}(T) = \sum_{i \in T} v(\{i\})$ for

each $T \in 2^S \setminus \{S\}$, and $v_S^{PSh}(S) = \sum_{i \in S} PSh_i(N, v)$. As a consequence, for each $i \in S$,

$$\begin{aligned}
 PSh_i(S, v_S^{PSh}) &= \frac{v_S^{PSh}(\{i\})}{\sum_{j \in N} v_S^{PSh}(\{j\})} v_S^{PSh}(S) \\
 &= \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \sum_{k \in S} PSh_k(N, v) \\
 &= \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \sum_{k \in S} \frac{v(\{k\})}{\sum_{j \in N} v(\{j\})} v(N) \\
 &= \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) \\
 &= PSh_i(N, v).
 \end{aligned}$$

Finally, since any two-player game in \mathcal{C}^0 is quasi-additive, PSh satisfies **PS** by applying (3.5) to the two-player case. \blacksquare

Building on Propositions 3.5 and 3.6, we offer two characterizations of PSh that are comparable to two new characterizations of Sh in the sense that they only differ with respect to one axiom. Both characterizations have in common the three axioms in Proposition 3.5: **E**, **DPO** and **WL**. The first one invokes **WC**.

Proposition 3.7. *A value f on \mathcal{C}^0 satisfies **E**, **DPO**, **WL**, **WC** and*

- (i) **PS** if and only if $f = PSh$;
- (ii) **S** if and only if $f = Sh$.

Proof. By Proposition 3.6 and the fact that Sh satisfies **E**, **DPO**, **WL**, **WC** and **S**, it suffices to show the uniqueness parts. Consider any value f on \mathcal{C}^0 that satisfies **E**, **DPO**, **WL** and **WC**. By Proposition 3.5, it only remains to show that f is uniquely determined on \mathcal{QA}^0 . Pick any quasi-additive game $(N, v) \in \mathcal{QA}^0$, which means that $v = v_a + bu_N$ for some $a \in \mathbb{R}_{++}^N$ or $-a \in \mathbb{R}_{++}^N$ and $b \in \mathbb{R}$. In case $b = 0$, $v = v_a$ so that (N, v) is additive. All players are dummy, which implies that $f(N, v_a)$ is completely determined by combining **E** and **DPO**. In case $b \neq 0$, we distinguish two cases.

Firstly, assume as in point (i) that f satisfies **PS**. The proof borrows some steps of the proof of Theorem B in Hart and Mas-Colell (1989). We show that f admits a proportional potential on \mathcal{QA}^0 . To do so, define the function R for games $(N, v) \in \mathcal{QA}^0$ with at most two players by setting $R(\emptyset, v) = 0$, $R(\{i\}, v) =$

$v(\{i\})/|v(\{i\})|$, and

$$R(\{i, j\}, v) = \frac{v(\{i, j\}) + v(\{i\}) + v(\{j\})}{|v(\{i\}) + v(\{j\})|}.$$

By **E** and **PS**, for all $(N, v) \in \mathcal{QA}^0$ such that $n \leq 2$, and all $i \in N$, it holds that

$$f_i(N, v) = |v(\{i\})|(R(N, v) - R(N \setminus \{i\}, v)). \quad (3.6)$$

We now prove, by induction on the size of the player set, that R can be extended to all games in \mathcal{QA}^0 , i.e. that R is the proportional potential Q on \mathcal{QA}^0 , and in turn that $f = PSh$ on \mathcal{QA}^0 .

INITIALIZATION. As noted before, the assertion holds for games $(N, v) \in \mathcal{QA}^0$ with $n \leq 2$.

INDUCTION HYPOTHESIS. Assume that R has been defined and satisfies (3.6) for all games $(N, v) \in \mathcal{QA}^0$ such that $n \leq q$, $q \geq 2$.

INDUCTION STEP. Consider any $(N, v) \in \mathcal{QA}^0$ with $n = q + 1$. We have to show that $f_i(N, v)/|v(\{i\})| + R(N \setminus \{i\}, v)$ is independent of $i \in N$. Pick a triple of distinct players, which is always possible since $n \geq 3$. By Lemma 3.2 (since **DPO** and **E** imply **IGP**), **WC** and (3.6), we can write:

$$\begin{aligned} \frac{f_i(N, v)}{|v(\{i\})|} - \frac{f_j(N, v)}{|v(\{j\})|} &= \frac{f_i(N \setminus \{k\}, v_{N \setminus \{k\}}^f)}{|v(\{i\})|} - \frac{f_j(N \setminus \{k\}, v_{N \setminus \{k\}}^f)}{|v(\{j\})|} \\ &= \frac{f_i(N \setminus \{k\}, v_{N \setminus \{k\}}^f)}{|v_{N \setminus \{k\}}^f(\{i\})|} - \frac{f_j(N \setminus \{k\}, v_{N \setminus \{k\}}^f)}{|v_{N \setminus \{k\}}^f(\{j\})|} \\ &= (R(N \setminus \{k\}, v_{N \setminus \{k\}}^f) - R(N \setminus \{i, k\}, v_{N \setminus \{k\}}^f)) \\ &\quad - (R(N \setminus \{k\}, v_{N \setminus \{k\}}^f) - R(N \setminus \{j, k\}, v_{N \setminus \{k\}}^f)) \\ &= (R(N \setminus \{j, k\}, v_{N \setminus \{k\}}^f) - R(N \setminus \{i, j, k\}, v_{N \setminus \{k\}}^f)) \\ &\quad - (R(N \setminus \{i, k\}, v_{N \setminus \{k\}}^f) - R(N \setminus \{i, j, k\}, v_{N \setminus \{k\}}^f)). \end{aligned}$$

Another application of **WC** and two applications of (3.6) yield that the preceding equality becomes:

$$\begin{aligned}
&= \frac{f_i(N \setminus \{j, k\}, v_{N \setminus \{k\}}^f)}{|v_{N \setminus \{k\}}^f(\{i\})|} - \frac{f_j(N \setminus \{i, k\}, v_{N \setminus \{k\}}^f)}{|v_{N \setminus \{k\}}^f(\{j\})|} \\
&= \frac{f_i(N \setminus \{j, k\}, v_{N \setminus \{k\}}^f)}{|v(\{i\})|} - \frac{f_j(N \setminus \{i, k\}, v_{N \setminus \{k\}}^f)}{|v(\{j\})|} \\
&= \frac{f_i(N \setminus \{j\}, v)}{|v(\{i\})|} - \frac{f_j(N \setminus \{i\}, v)}{|v(\{j\})|} \\
&= (R(N \setminus \{j\}, v) - R(N \setminus \{i, j\}, v)) \\
&\quad - (R(N \setminus \{i\}, v) - R(N \setminus \{i, j\}, v)) \\
&= R(N \setminus \{j\}, v) - R(N \setminus \{i\}, v)
\end{aligned}$$

as desired. Remark 3.1 points out that one-player reduced games of a quasi-additive game in \mathcal{QA}^0 may not belong to \mathcal{QA}^0 . In case $n = 3$, $(N \setminus \{j, k\}, v_{N \setminus \{k\}}^f)$ and $(N \setminus \{j, k\}, v_{N \setminus \{k\}}^f)$ are one-player games, but they both belong to \mathcal{QA}^0 . To see this, $(N, v) \in \mathcal{QA}^0$ and $n = 3$ imply that $(N \setminus \{k\}, v_{N \setminus \{k\}}^f) \in \mathcal{QA}^0$ by Lemma 3.2, and thus that for each nonempty $S \in 2^{N \setminus \{k\}}$, $(S, v_{N \setminus \{k\}}^f) \in \mathcal{QA}^0$ too.

Secondly, assume as in point (ii) that f satisfies **S**. The result follows from Hart and Mas-Colell (1989, Theorem B), where the preceding steps are developed on the basis of **S** and the classical potential function. ■

The logical independence of the axioms in Proposition 3.7 is demonstrated in appendix. The second result in this section compares once again the Shapley value and PSh by keeping axioms **E**, **DPO** and **WL**, and by adding either **PAM** or the following axiom.

Equal aggregate monotonicity (EAM). For all $b \in \mathbb{R}$, all $(N, v) \in \mathcal{C}^0$, and all $i, j \in N$,

$$f_i(N, v) - f_i(N, v + bu_N) = f_j(N, v) - f_j(N, v + bu_N).$$

Replacing **WC** and **PS** (resp. **S**) by **PAM** (resp. **EAM**) in Proposition 3.7 yields a characterization of PSh (resp. Sh) on \mathcal{C}^0 .

Proposition 3.8. *A value f on \mathcal{C}^0 satisfies **E**, **DPO**, **WL**, and*

- (i) **PAM** if and only if $f = PSh$;

(ii) **EAM** if and only if $f = Sh$.

Proof. Regarding point (i), by Proposition 3.6, it suffices to show uniqueness. Consider any value f on \mathcal{C}^0 that satisfies **E**, **DPO**, **WL** and **PAM**. By Proposition 3.5, it only remains to show that f is uniquely determined on \mathcal{QA}^0 . Pick any quasi-additive game $(N, v) \in \mathcal{QA}^0$, which means that $v = v_a + bu_N$ for some $a \in \mathbb{R}_{++}^N$ or some $-a \in \mathbb{R}_{++}^N$ and $b \in \mathbb{R}$. In case $b = 0$, $v = v_a$ is an additive function. All players are dummy, which implies $f(N, v_a)$ is completely determined by **E** and **DPO**. If $b \neq 0$, remark that (N, v) with $v = v_a + bu_N$ and (N, v_a) only differ with respect to the worth of the grand coalition N . By **PAM**, we have, for all $i, j \in N$,

$$\frac{f_i(N, v) - f_i(N, v_a)}{v(\{i\})} = \frac{f_j(N, v) - f_j(N, v_a)}{v(\{j\})}.$$

Summing on all $j \in N$ and using **E** in both games, we get

$$f_i(N, v) = f_i(N, v_a) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} (v(N) - v_a(N)) = f_i(N, v_a) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} b,$$

for all $i \in N$, and so $f_i(N, v)$ is uniquely determined, as desired.

Regarding point (ii), it is easy to check that Sh satisfies **EAM**. For the uniqueness part, mimics the proof of point (i) except in the very last part where the combination **EAM** and **E** implies that $f_i(N, v) = f_i(N, v_a) + b/n$ for all $i \in N$. ■

The logical independence of the axioms in Proposition 3.8 is demonstrated in appendix. It should be noted that replacing either **PS** or **S** in Proposition 3.7 or either **PAM** and **EAM** in Proposition 3.8 by Aggregate monotonicity (Megiddo, 1974) does not yield the set of (positively) weighted Shapley values. The weighted Shapley values satisfy all axioms, but they are not the only one. In fact, consider a value f on \mathcal{C}^0 such that, for any $N \in U$, any $a \in \mathbb{R}_{++}^N$ and any $(N, v) \in \mathcal{C}_N^a$ it holds that $f(N, v) = Sh^w(N, v)$ for some weights w . Whenever two games belonging to disjoint sets \mathcal{C}_N^a are associated to different weights, the value f satisfies all axioms but is not a (positively) weighted Shapley value.

As a final remark, it is worth noting that strengthening, for each $N \in U$, **WL** on \mathcal{C}_N^0 by **L** on \mathcal{C}_N in Proposition 3.8 (ii) yields a characterization of the Shapley value on the full domain \mathcal{C} by **E**, **DPO**, **WL**, and **EAM**. In case **WC** is further invoked on \mathcal{QA} instead of \mathcal{QA}^0 , then from Proposition 3.7 (ii), the Shapley value

can be characterized on the full domain \mathcal{C} by this axiom together with **E**, **DPO**, **WL**, and **S**.

3.4.2 Proportional Balanced contributions under dummification

Contrary to the results in section 3.4, we illustrate here that *PSH* can be characterized on a fixed player set by means of the following definition. For a game $(N, v) \in \mathcal{C}^0$ and a player $i \in N$, we denote by $(N, v^i) \in \mathcal{C}^0$ the game obtained from (N, v) if player i is **dummified**: $v^i(S) = v(\{i\}) + v(S \setminus \{i\})$ for all $S \in 2^N$ such that $S \ni i$ and $v^i(S) = v(S)$ for all $S \not\ni i$.⁶ The dummification operation is similar to the nullification operation studied in Béal et al. (2016), among others. The dummification arises naturally in the so-called Myerson (graph) restricted game (Myerson, 1977), where, for a given graph on the player set, the worth of a coalition is the sum of the worths of its connected parts. If a player is deprived of his or her links then he or she becomes dummified in the resulting new Myerson restricted game. Below, we introduce a variant of **PBC** in which the subgame induced when a player leaves the game is replaced by the game in which this player is dummified.

Remark 3.2. Note that $(v^i)^i = v^i$ and $(v^i)^j = (v^j)^i$. From the latter property, for each nonempty $S \in 2^N$, the function v^S in which the players in S are (successively) dummified is well-defined. For any $(N, v) \in \mathcal{C}_N^0$ and $i \in N$, $v^{N \setminus \{i\}} = v^N$ and $(N, v^{N \setminus \{i\}}) \in \mathcal{A}_N^0$, i.e. this game is additive. Regarding the set of dummy players $D(N, v)$, note also that $D(N, v) = \{i \in N : v^i = v\}$, and that for any $(N, v) \in \mathcal{C}_N^0$ and $i \in N$, $D(N, v^i) \supseteq D(N, v) \cup \{i\}$, where this inclusion may be strict. \square

Proportional balanced contributions under dummification, PBCD. For all $(N, v) \in \mathcal{C}_N^0$, all $i, j \in N$,

$$\frac{f_i(N, v) - f_i(N, v^j)}{v(\{i\})} = \frac{f_j(N, v) - f_j(N, v^i)}{v(\{j\})}.$$

For a fixed player set N , Proposition 3.9 below indicates that a value satisfying **E** and **PBCD** is completely determined by what it prescribes on additive games with player set N .

⁶An equivalent, and perhaps shorter, definition is that $v^i(S) = v(S \setminus \{i\}) + v(S \cap \{i\})$ for all $S \in 2^N$.

Proposition 3.9. Consider two values f and g satisfying **E** and **PBCD** on \mathcal{C}_N^0 such that $f = g$ on \mathcal{A}_N^0 . Then $f = g$ on \mathcal{C}_N^0 .

Proof. Consider two values f and g satisfying **E** and **PBCD** on \mathcal{C}_N^0 such that $f = g$ on \mathcal{A}_N^0 . The proof that $f = g$ on \mathcal{C}_n^0 is done by (descending) induction on the number of dummy players.

INITIALIZATION. For a game $(N, v) \in \mathcal{C}_N^0$, if $|D(N, v)| = n$, i.e. all players are dummy. Then v is additive and $f = g$ by hypothesis. By Remark 3.2, there is no game in which $|D(N, v)| = n - 1$.

INDUCTION HYPOTHESIS. Assume that $f(N, v) = g(N, v)$ for all games $(N, v) \in \mathcal{C}_N^0$ such that $|D(N, v)| \geq d$, $0 < d \leq n - 1$.

INDUCTION STEP. Choose any game $(N, v) \in \mathcal{C}_N^0$ such that $|D(N, v)| = d - 1$. Because $|D(N, v)| < n - 1$, there exists $i \in N \setminus D(N, v)$, which implies $D(N, v^i) \supseteq D(N, v) \cup \{i\}$ and $|D(N, v^i)| \geq |D(N, v)| + 1 = d$. Now pick any $j \in D(N, v)$. It holds that $v = v^j$, so that **PBCD** and the induction hypothesis imply that

$$\begin{aligned} f_j(N, v) &= f_j(N, v^i) + \frac{v(\{j\})}{v(\{i\})} (f_i(N, v) - f_i(N, v^j)) \\ &= f_j(N, v^i) \\ &= g_j(N, v^i) \\ &= g_j(N, v^i) + \frac{v(\{j\})}{v(\{i\})} (g_i(N, v) - g_i(N, v^j)) \\ &= g_j(N, v). \end{aligned} \tag{3.7}$$

Conclude that the assertion is proved for dummy players in (N, v) . Next, pick any $j \in N \setminus (D(N, v) \cup \{i\})$. Note that $N \setminus (D(N, v) \cup \{i\}) \neq \emptyset$ since $|D(N, v)| < n + 1$. Applied to i and j , **PBCD** can be rewritten as follows:

$$f_j(N, v) = f_j(N, v^i) + \frac{v(\{j\})}{v(\{i\})} f_i(N, v) - \frac{v(\{j\})}{v(\{i\})} f_i(N, v^j). \tag{3.8}$$

Reformulation (3.8) can be done for g too. Note that $|D(N, v^j)| \geq d$. Using **E** for f and g gives:

$$\begin{aligned} v(N) &= f_i(N, v) + \sum_{j \in D(N, v)} f_j(N, v) + \sum_{j \in N \setminus (D(N, v) \cup \{i\})} f_j(N, v) \\ v(N) &= g_i(N, v) + \sum_{j \in D(N, v)} g_j(N, v) + \sum_{j \in N \setminus (D(N, v) \cup \{i\})} g_j(N, v) \end{aligned}$$

Subtracting the lower equation to the upper, using (3.7), (3.8) and the induction hypothesis yield:

$$\begin{aligned}
0 &= f_i(N, v) - g_i(N, v) + \sum_{j \in N \setminus (D(N, v) \cup \{i\})} [f_j(N, v^i) - g_j(N, v^i) \\
&\quad + \frac{v(\{j\})}{v(\{i\})} (f_i(N, v) - g_i(N, v)) - \frac{v(\{j\})}{v(\{i\})} (f_i(N, v^j) - g_i(N, v^j))] \\
&= (f_i(N, v) - g_i(N, v)) \times \left(1 + \sum_{j \in N \setminus (D(N, v) \cup \{i\})} \frac{v(\{j\})}{v(\{i\})} \right) \quad (3.9)
\end{aligned}$$

Since $(N, v) \in \mathcal{C}_N^0$, the right term in (3.9) is positive, and so $f_i(N, v) = g_i(N, v)$ for any non-dummy player $i \in N \setminus D(N, v)$. This completes the proof. ■

In order to characterize PSh , we invoke **IGP**.

Proposition 3.10. *The proportional Shapley value is the unique value on \mathcal{C}_N^0 that satisfies **E**, **PBCD** and **IGP**.*

Proof. Clearly, PSh satisfies the three axioms. So consider any value f on \mathcal{C}_N^0 satisfying the three axioms. By **IGP**, f is uniquely determined on \mathcal{A}_N^0 . Since f also satisfies **E** and **PBCD**, Proposition 3.9 implies that f is also uniquely determined on \mathcal{C}_N^0 . ■

3.5 Conclusion

The promising results obtained for the land production economies reveal that the proportional Shapley value can outperform the (weighted) Shapley value(s) in specific cases. A challenging extension of our work would be to confirm or invalidate this assessment by study the proportional Shapley value in the other applications listed in the introduction of the article. Finally, let us conclude by mentioning a recurrent weakness related to weighted values. Haeringer (2006) argue that some information is contained in the Harsanyi dividends since it can be interpreted as the coalitions' contribution to the worth of the grand coalition. He advocates that the distribution of a Harsanyi dividends among its members should depend on its sign, i.e. whether the associated coalition contributes negatively or positively to the worth of the grand coalition. Coming back to the proportional Shapley value, in a

game with negative stand-alone worths (which is not so common in applications), positive dividends are distributed in inverse proportion to these stand-alone worths: the players with the worse stand-alone worth get the best shares of the dividend. This difficulty may be overcome by considering the approach developed in [Haeringer \(2006\)](#), thus ensuring that the associated payoff to a player is always increasing with respect to his or her initial weight. This is left for future work.

Appendix

Land production economies

We start by stating a Lemma which is essential to prove Proposition 3.1.

Lemma 3.3. *For any land production economy (N, a, z) , there exists a unique function $g : N \rightarrow \mathbb{R}$ such that $\Delta_{v_a, z} = v_{a, g}$, and defined, for each $s = \{1, \dots, n\}$, by:*

$$g(s) = \sum_{k=0}^{s-1} (-1)^{s-1-k} \binom{s-1}{k} z(k+1). \quad (3.10)$$

Moreover, for each $s = \{1, \dots, n\}$, it holds that:

$$z(s) = \sum_{k=0}^{s-1} \binom{s-1}{k} g(k+1). \quad (3.11)$$

Proof. Consider any land production economy (N, a, z) . For each $S \in 2^N$, we have:

$$\begin{aligned} \Delta_{v_a, z}(S) &= \sum_{T \subseteq S} (-1)^{s-t} v_{a, z}(T) \\ &= \sum_{T \subseteq S} (-1)^{s-t} z(t) \frac{1}{L} \sum_{i \in T} a_i \\ &= \frac{1}{L} \sum_{i \in S} a_i \sum_{T \subseteq S, T \ni i} (-1)^{s-t} z(t) \\ &= \frac{1}{L} \sum_{i \in S} a_i \underbrace{\sum_{k=0}^{s-1} (-1)^{s-1-k} \binom{s-1}{k} z(k+1)}_{g(s)}. \end{aligned}$$

The last equation defines g and we have $\Delta_{v_{a,z}} = v_{a,g}$. Conversely, we may recover z :

$$\begin{aligned}
 v_{a,z}(S) &= \sum_{T \subseteq S} \Delta_{v_{a,z}}(S) \\
 &= \sum_{T \subseteq S} g(t) \frac{1}{L} \sum_{i \in T} a_i \\
 &= \frac{1}{L} \sum_{i \in S} a_i \sum_{T \subseteq S, T \ni i} g(t) \\
 &= \frac{1}{L} \sum_{i \in S} a_i \underbrace{\sum_{k=0}^{s-1} \binom{s-1}{k} g(k+1)}_{=z(s)},
 \end{aligned}$$

which completes the proof. ■

Proof. (Proposition 3.1) Consider any land production economy (N, a, z) . For each $i \in N$, by Lemma 3.3, we obtain:

$$\begin{aligned}
 PSh_i(N, v_{a,z}) &= \sum_{S \in 2^N: S \ni i} \frac{v_{a,z}(\{i\})}{\sum_{j \in S} v_{a,z}(\{j\})} \Delta_{v_{a,z}}(S) \\
 &= \sum_{S \in 2^N: S \ni i} \frac{a_i}{\sum_{j \in S} a_j} \left(g(s) \frac{1}{L} \sum_{k \in S} a_k \right) \\
 &= a_i \frac{1}{L} \sum_{S \in 2^N: S \ni i} g(s) \\
 &= a_i \frac{1}{L} \sum_{k=0}^{n-1} \binom{n-1}{k} g(k+1) \\
 &= \frac{a_i}{L} z(n),
 \end{aligned}$$

as desired. ■

Next, we provide a formulation of the Shapley value for land production economies. To this end, we rely on generating functions (see chapter 4 in [Stanley, 1986](#), for an introduction), which have been widely used to compute power indices, for instance in [Alonso-Mejide et al. \(2014\)](#). For any function $z : N \rightarrow \mathbb{R}$, let us define the (exponential) generating functions $Z(x) = \sum_{k \geq 0} z(k+1) x^k / k!$ and the corresponding $G(x) = \sum_{k \geq 0} g(k+1) x^k / k!$, where $g : N \rightarrow \mathbb{R}$ is defined by formula (3.10). Here too, a Lemma is useful.

Lemma 3.4. *For any $z : N \rightarrow \mathbb{R}$, one has $G(x) = e^{-x} Z(x)$.*

Proof. Let us show that formula (3.11) is translated into $Z(x) = e^x G(x)$ in the context of exponential generating functions:

$$\begin{aligned}
 e^x G(x) &= \left(\sum_{l \geq 0} \frac{x^l}{l!} \right) \times \left(\sum_{k \geq 0} g(k+1) \frac{x^k}{k!} \right) \\
 &= \sum_{k, l \geq 0} g(k+1) \frac{x^{l+k}}{l!k!} \\
 &= \sum_{n \geq 0} \sum_{k=0}^n g(k+1) \frac{x^n}{(n-k)!k!} \\
 &= \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k} g(k+1) \right) \frac{x^n}{n!} \\
 &= \sum_{n \geq 0} z(n+1) \frac{x^n}{n!} \\
 &= Z(x),
 \end{aligned} \tag{3.12}$$

as desired. ■

For two exponential generating functions Z_1, Z_2 , define the convolution operation $(Z_1 \star Z_2)(x) = \int_0^x Z_1(t) Z_2(x-t) dt$. Recall that $(Z_1 \star Z_2)'(x) = (Z_1 \star Z_2')(x) + Z_1(x) Z_2(0)$ for instance.

Proposition 3.11. *For any land production economy (N, a, z) , it holds that:*

$$Sh_i(N, v_{a,z}) = \frac{1}{L} \left(s(n) a_i + h(n) \sum_{j \in N \setminus i} a_j \right) \tag{3.13}$$

where function s comes from the exponential generating function $S(x) = \sum_{k \geq 0} s(k+1) x^k / k! = (Z \star \exp)(x) / x$ and $H(x) = \sum_{k \geq 0} h(k+2) x^k / k! = (xZ(x) - (Z \star \exp)(x)) / x^2$.

Proof. Firstly, let us show that the Shapley value may be written as (3.13):

$$\begin{aligned}
Sh_i(N, v_{a,z}) &= \sum_{S \ni i} \frac{\Delta_{v_{a,z}}(S)}{s} \\
&= \sum_{S \ni i} \frac{v_{a,g}(S)}{s} \\
&= \sum_{S \ni i} \left(\frac{g(s)}{s} \frac{1}{L} \sum_{j \in S} a_j \right) \\
&= \frac{1}{L} \sum_{j=1}^n a_j \left(\sum_{S \ni i, j} \frac{g(s)}{s} \right) \\
&= \frac{1}{L} \left(\left(\sum_{S \ni i} \frac{g(s)}{s} \right) a_i + \sum_{j \in N \setminus i} a_j \left(\sum_{S \ni i, j} \frac{g(s)}{s} \right) \right) \\
&= \frac{1}{L} \left(\left(\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{g(k+1)}{k+1} \right) a_i + \sum_{j \in N \setminus i} a_j \left(\sum_{k=0}^{n-2} \binom{n-2}{k} \frac{g(k+2)}{k+2} \right) \right) \\
&= \frac{1}{L} \left(\underbrace{\left(\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{g(k+1)}{k+1} \right)}_{s(n)} a_i + \sum_{j \in N \setminus i} a_j \underbrace{\left(\sum_{k=0}^{n-2} \binom{n-2}{k} \frac{g(k+2)}{k+2} \right)}_{h(n)} \right)
\end{aligned} \tag{3.14}$$

Secondly, having defined $S(x) = \sum_{k \geq 0} s(k+1)x^k/k!$, let us connect $S(x)$ with $Z(x)$.

$$\begin{aligned}
 S(x) &= \sum_{n \geq 0} s(n+1) \frac{x^n}{n!} \\
 &= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} \frac{g(k+1)}{k+1} \frac{x^n}{n!} \\
 &= \sum_{n \geq 0} \sum_{k=0}^n \left(\frac{g(k+1)}{k+1} \frac{x^k}{k!} \right) \left(\frac{x^{n-k}}{(n-k)!} \right) \\
 &= \sum_{k, l \geq 0} \left(\frac{g(k+1)}{k+1} \frac{x^k}{k!} \right) \frac{x^l}{l!} \\
 &= \left(\sum_{k \geq 0} g(k+1) \frac{x^k}{k+1} \frac{1}{k!} \right) \times \left(\sum_{l \geq 0} \frac{x^l}{l!} \right) \tag{3.15}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\sum_{k \geq 0} g(k+1) \frac{1}{x} \int_0^x t^k dt \frac{1}{k!} \right) \times e^x \\
 &= \frac{e^x}{x} \int_0^x \left(\sum_{k \geq 0} g(k+1) \frac{t^k}{k!} \right) dt \\
 &= \frac{e^x}{x} \int_0^x G(t) dt \\
 &= \frac{e^x}{x} \int_0^x Z(t) e^{-t} dt \\
 &= \frac{1}{x} \int_0^x Z(t) e^{x-t} dt \\
 &= \frac{(Z \star \exp)(x)}{x} \tag{3.16}
 \end{aligned}$$

Lastly, we find a closed expression for $H(x) = \sum_{k \geq 0} h(k+2)x^k/k!$. We will need to define $\hat{G}(x) = \sum_{k \geq 0} g(k+1)/(k+1) \times (x^k/k!)$ so that by (3.15), $\hat{G}(x) = S(x)e^{-x}$.

Moreover $\hat{G}'(x) = \sum_{k \geq 0} g(k+2)/(k+2) \times (x^k/k!)$.

$$\begin{aligned}
H(x) &= \sum_{n \geq 0} h(n+2) \frac{x^n}{n!} \\
&= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} \frac{g(k+2)}{k+2} \frac{x^n}{n!} \\
&= \left(\sum_{k \geq 0} g(k+2) \frac{x^k}{k+2} \frac{1}{k!} \right) \times \left(\sum_{l \geq 0} \frac{x^l}{l!} \right) \\
&= e^x \left(\sum_{k \geq 0} g(k+2) \frac{x^k}{k+2} \frac{1}{k!} \right) \\
&= e^x \hat{G}'(x) \\
&= e^x (S'(x)e^{-x} - S(x)e^{-x}) \\
&= S'(x) - S(x) \\
&= \frac{x(F \star \exp)(x) + xZ(x) \exp(0) - (Z \star \exp)(x)}{x^2} - \frac{(Z \star \exp)(x)}{x} \\
&= \frac{xZ(x) - (Z \star \exp)(x)}{x^2} \tag{3.17}
\end{aligned}$$

■

Below are some examples of specification of function z .

- if $z(s) = 1$, then $Z(x) = e^x$ so that $G(x) = 1$. Moreover $S(x) = e^x$ and $H(x) = 0$.
- if $z(s) = s$, then $Z(x) = (1+x)e^x$ so that $G(x) = 1+x$. Moreover $S(x) = e^x(1+x/2)$ and $H(x) = e^x/2$.
- if $z(s) = 1/s$, then $Z(x) = (e^x - 1)/x$ so that $G(x) = (1 - e^{-x})/x = F(-x)$.
- if $z(s) = 2^{s-1}$, then $Z(x) = e^{2x}$ so that $G(x) = e^x$. Moreover $S(x) = e^x(e^x - 1)/x$ and $H(x) = e^{2x}(e^{-x} + x - 1)/x^2$.

Proof of Proposition 3.5

Throughout this section, we consider fixed $N \in U$ and $a \in \mathbb{R}_{++}^N$. Several definitions will be useful. Firstly, define the class \mathcal{C}_N^{a+} as

$$\mathcal{C}_N^{a+} = \{(N, v) \in \mathcal{C}_N \mid \exists c \in \mathbb{R} : \forall i \in N, v(\{i\}) = ca_i\},$$

so that $\mathcal{C}_N^a = \mathcal{C}_N^{a+} \cap \mathcal{C}_N^0$, i.e. \mathcal{C}_N^a contains all games in \mathcal{C}_N^{a+} , except those with null stand-alone worths. Obviously, \mathcal{C}_N^{a+} is a real vector space. Furthermore, the dimension of \mathcal{C}_N^{a+} is $2^n - 1 - n + 1 = 2^n - n$. Note also that \mathcal{C}_N^{a+} is the smallest vector space that contains \mathcal{C}_N^a . Secondly, for all $S \in 2^N$ such that $s \geq 2$, define the game (N, r_S) such that $r_S = u_S + v_a$. Lemma 3.5 essentially states that all games in \mathcal{C}_N^a admit a unique decomposition via the collection $\{(N, v_a), (N, r_S)_{S \in 2^N: s \geq 2}\}$ and enunciates properties of the associated coefficients. More specifically, $\{(N, v_a), (N, r_S)_{S \in 2^N: s \geq 2}\}$ is a basis for the vector space \mathcal{C}_N^{a+} , and this basis is composed of games in \mathcal{C}_N^a only.

Lemma 3.5. *Consider any $(N, v) \in \mathcal{C}_N^{a+}$, and let $v(\{i\}) = ca_i$ for all $i \in N$, $c \in \mathbb{R}$. Then,*

(i) *there are unique coefficients $\gamma_v(S) \in \mathbb{R}$, $S \in 2^N$, $s \geq 2$, and $\gamma_v(0) \in \mathbb{R}$ such that*

$$v = \sum_{S \in 2^N: s \geq 2} \gamma_v(r_S) r_S + \gamma_v(v_a) v_a;$$

(ii) *for all $S \in 2^N$, $s \geq 2$, $\gamma_v(r_S) = \Delta_{v-cv_a}(S)$, and $\gamma_v(v_a) = c - \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S)$;*

(iii) $\sum_{S \in 2^N: s \geq 2} \gamma_v(r_S) + \gamma_v(v_a) = c$;

(iv) $(N, v) \in \mathcal{C}_N^a$ if and only if $\sum_{S \in 2^N: s \geq 2} \gamma_v(r_S) + \gamma_v(v_a) \neq 0$.

Proof. Consider any $(N, v) \in \mathcal{C}_N^{a+}$, and let $v(\{i\}) = ca_i$ for all $i \in N$, $c \in \mathbb{R}$. We can write v as $(v - cv_a) + cv_a$, where $(v - cv_a)$ is a characteristic function on N that vanishes for singletons. Therefore, $v - cv_a$ can be written as

$$(v - cv_a) = \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) u_S.$$

From this, we get

$$\begin{aligned} v &= (v - cv_a) + cv_a \\ &= \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) u_S + cv_a \\ &= \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) (u_S + v_a) - \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) v_a + cv_a \\ &= \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) (u_S + v_a) + \left(c - \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) \right) v_a \\ &= \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) r_S + \left(c - \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S) \right) v_a \end{aligned} \quad (3.18)$$

Letting $\gamma_v(r_S) = \Delta_{v-cv_a}(S)$ and $\gamma_v(v_a) = c - \sum_{S \in 2^N: s \geq 2} \Delta_{v-cv_a}(S)$, we obtain

$$v = \sum_{S \in 2^N: s \geq 2} \gamma_v(r_S) r_S + \gamma_v(v_a) v_a.$$

So, the collection of games $\{(N, v_a), (N, r_S)_{S \in 2^N: s \geq 2}\}$ spans \mathcal{C}_N^{a+} , and this collection contains $2^n - n$ elements, i.e. as many elements as the dimension of \mathcal{C}_N^{a+} . Conclude that $\{(N, v_a), (N, r_S)_{S \in 2^N: s \geq 2}\}$ is a basis for the vector space \mathcal{C}_N^{a+} . Therefore, any game $(N, v) \in \mathcal{C}_N^{a+}$ is uniquely decomposed as in (3.18), proving claim (i). Claim (ii) follows from (3.18), claim (iii) is obvious via claim (ii), and claim (iv) is obvious from claim (iii). ■

Lemma 3.6 is technical and will be used on the coefficients exhibited in Lemma 3.5 (i) so as to ensure the property highlighted in Lemma 3.5 (iv).

Lemma 3.6. *Let (x_1, \dots, x_q) be a sequence of $q \geq 1$ real numbers such that $x_k \in \mathbb{R}^*$ for all $k \in \{1, \dots, q\}$ and $\sum_{k=1}^q x_k \in \mathbb{R}^*$. Then, there exists an ordering $(x_{(1)}, \dots, x_{(q)})$ of (x_1, \dots, x_q) such that, for all $k \in \{1, \dots, q\}$, $\sum_{l=1}^k x_{(l)} \in \mathbb{R}^*$.*

Proof. The proof is by induction on q .

INITIALIZATION. The claim is trivial for the case $q = 1$, and any of the two possible orderings can be used to prove easily the case $q = 2$.

INDUCTION HYPOTHESIS. Assume that there exists a desired ordering for all allowed sequences (x_1, \dots, x_q) such that $q \leq \bar{q}$, $\bar{q} \geq 2$.

INDUCTION STEP. Consider a sequence (x_1, \dots, x_q) , $q = \bar{q} + 1$, such that $x_k \in \mathbb{R}^*$ for all $k \in \{1, \dots, q\}$ and $\sum_{k=1}^q x_k \in \mathbb{R}^*$. We distinguish two cases. Firstly, suppose that $\sum_{k=1}^{q-1} x_k \in \mathbb{R}^*$, then the induction hypothesis can be applied to the sub-sequence (x_1, \dots, x_{q-1}) . The desired ordering on (x_1, \dots, x_q) is constructed by considering a desired ordering on the sub-sequence (x_1, \dots, x_{q-1}) and by adding number x_q in position q . Secondly, suppose that $\sum_{k=1}^{q-1} x_k = 0$. Since the numbers x_k , $k \in \{1, \dots, q-1\}$, are all non-null, there exists a number x_k , $k \in \{1, \dots, q-1\}$, such that $\text{sign}(x_k) = -\text{sign}(x_q)$. Thus $\sum_{l \in \{1, \dots, k-1, k+1, \dots, q\}} x_l \in \mathbb{R}^*$, which means that the induction hypothesis can be applied to the sub-sequence $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_q)$. Similarly as before, the desired ordering on (x_1, \dots, x_q) is constructed by considering a desired ordering on the sub-sequence $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_q)$ and by adding number x_k in position q . ■

Proof. (Proposition 3.5) We shall show that if a value on \mathcal{C}^0 satisfies **E**, **DPO** and **WL**, and is uniquely determined on \mathcal{QA}^0 , then f is uniquely determined on \mathcal{C}^0 . So consider such a value f . Fix some $N \in \mathcal{U}$ and some $a \in \mathbb{R}_{++}^N$. Pick any $(N, v) \in \mathcal{C}_N^a$. By Lemma 3.5, we have that

$$v = \sum_{S \in 2^N: s \geq 2} \gamma_v(r_S) r_S + \gamma_v(v_a) v_a.$$

Let $q \in \{1, \dots, 2^n - n\}$ be the number of non-null coefficients in the above decomposition, where $q > 0$ by definition of \mathcal{C}_N^a and Lemma 3.5 (iii). Denote by $(\gamma_v(v_1), \dots, \gamma_v(v_q))$ the associated sequence of coefficients. By Lemma 3.5 (iii), it holds that $\sum_{k=1}^q \gamma_v(v_k) = c \neq 0$. Thus, we can apply Lemma 3.6: there is an ordering $(\gamma_v(v_{(1)}), \dots, \gamma_v(v_{(q)}))$ of $(\gamma_v(v_1), \dots, \gamma_v(v_q))$ such that, for all $k \in \{1, \dots, q\}$,

$$\sum_{l=1}^k \gamma_v(v_{(l)}) \neq 0. \quad (3.19)$$

Now, denote by (N, v^k) the game such that

$$v^k = \sum_{l=1}^k \gamma_v(v_{(l)}) v_{(l)}.$$

By (3.19) and Lemma 3.5 (iv), $(N, v^k) \in \mathcal{C}_N^a$. Successive applications of **WL** to games (N, v^k) and $(N, \gamma_v(v_{k+1}) v_{k+1})$ for all $k \in \{1, \dots, q-1\}$ according to ordering $(\gamma_v(v_{(1)}), \dots, \gamma_v(v_{(q)}))$ imply that

$$f(N, v) = \sum_{S \in 2^N: s \geq 2} \gamma_v(r_S) f(N, r_S) + \gamma_v(v_a) f(N, v_a).$$

Note that $(N, v_a) \in \mathcal{QA}^0$. Moreover, consider any $S \in 2^N$ such that $s \geq 2$. Each player $i \in N \setminus S$ is dummy in (N, r_S) , which means that $f_i(N, r_S)$ is uniquely determined by **E** and **DPO** for such players. Then $n - s$ successive applications of **DPO** yield that $f_i(N, r_S) = f_i(S, r_S)$ for all $i \in S$. Remark that $(S, r_S) \in \mathcal{QA}^0$. By assumption f is uniquely determined on \mathcal{QA}^0 , so that f is uniquely determined in games (N, r_S) , $S \in 2^N$, $s \geq 2$, and (N, v_a) . This completes the proof. \blacksquare

Logical independence of the axioms in Propositions 3.7 and 3.8

Proposition 3.7:

- The Shapley value Sh on \mathcal{C}^0 satisfies **E**, **DPO**, **WL**, **WC** but not **PS**.
- PSh on \mathcal{C}^0 satisfies **E**, **DPO**, **WL**, **WC** but not **S**.
- The value on \mathcal{C}^0 which assigns to each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

$$\Delta_v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} \Delta_v(N)$$

satisfies **DPO**, **WL**, **WC**, **PS** but not **E**.

- The value on \mathcal{C}^0 which assigns to each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, the payoff $\Delta_v(\{i\}) + \Delta_v(N)/n$ satisfies **DPO**, **WL**, **WC**, **S** but not **E**.
- The proportional value on \mathcal{C}^0 , which assigns to each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

$$\frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$$

satisfies **E**, **WL**, **WC**, **PS** but not **DPO**.

- The equal surplus division ESD on \mathcal{C}^0 , which assigns to each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

$$ESD_i(N, v) = v(\{i\}) + \frac{1}{n} \left(v(N) - \sum_{j \in N} v(\{j\}) \right)$$

satisfies **E**, **WL**, **WC**, **PS**, but not **DPO**.

- The value f on \mathcal{C}^0 defined for each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, by

$$f_i(N, v) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} \Delta_v(N) + \sum_{S \in 2^N : S \ni i, S \in \{2, \dots, n-1\}} \frac{v(\{i\})^2}{\sum_{j \in S} v(\{j\})^2} \Delta_v(S).$$

satisfies **E**, **DPO**, **WC**, **PS** but not **WL**.

- The value f on \mathcal{C}^0 defined for each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, by

$$f_i(N, v) = v(\{i\}) + \frac{1}{n} \Delta_v(N) + \sum_{S \in 2^N: S \ni i, s \in \{2, \dots, n-1\}} \frac{v(\{i\})^2}{\sum_{j \in S} v(\{j\})^2} \Delta_v(S).$$

satisfies **E**, **DPO**, **WC**, **PS** but not **WL**.

- For a given integer $k \geq 2$, the value f on \mathcal{C}^0 defined for each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, by

$$f_i(N, v) = \sum_{S \in 2^N: S \ni i, s \leq k} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \sum_{S \in 2^N: S \ni i, s > k} \frac{\Delta_v(S)}{s}.$$

satisfies **E**, **DPO**, **WL**, **PS** but not **WC**.

- For a given integer $k \geq 2$, the value f on \mathcal{C}^0 defined for each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, by

$$f_i(N, v) = \sum_{S \in 2^N: S \ni i, s > k} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \sum_{S \in 2^N: S \ni i, s \leq k} \frac{\Delta_v(S)}{s}.$$

satisfies **E**, **DPO**, **WL**, **S** but not **WC**.

Proposition 3.8:

- The null solution on \mathcal{C}^0 satisfies **DPO**, **WL**, **PAM**, **EAM**, but not **E**.
- The Shapley value Sh on \mathcal{C}^0 satisfies **E**, **DPO**, **WL**, **EAM** but not **PAM**.
- PSh on \mathcal{C}^0 satisfies **E**, **DPO**, **WL**, **PAM** but not **EAM**.
- The proportional value on \mathcal{C}^0 , which assigns to each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

$$\frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$$

satisfies **E**, **WL**, **PAM**, but not **DPO**.

- The equal surplus division ESD on \mathcal{C}^0 , which assigns to each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

$$ESD_i(N, v) = v(\{i\}) + \frac{1}{n} \left(v(N) - \sum_{j \in N} v(\{j\}) \right)$$

satisfies **E**, **WL**, **EAM**, but not **DPO**.

- The value f on \mathcal{C}^0 defined for each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, by

$$f_i(N, v) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} \Delta_v(N) + \sum_{S \in 2^N : S \ni i, s \in \{2, \dots, n-1\}} \frac{v(\{i\})^2}{\sum_{j \in S} v(\{j\})^2} \Delta_v(S).$$

satisfies **E**, **DPO**, **PAM**, but not **WL**.

- The value f on \mathcal{C}^0 defined for each $(N, v) \in \mathcal{C}^0$ and each $i \in N$, by

$$f_i(N, v) = v(\{i\}) + \frac{1}{n} \Delta_v(N) + \sum_{S \in 2^N : S \ni i, s \in \{2, \dots, n-1\}} \frac{v(\{i\})^2}{\sum_{j \in S} v(\{j\})^2} \Delta_v(S).$$

satisfies **E**, **DPO**, **EAM** but not **WL**.

Bibliography

- Alonso-Meijide, J. M., Bilbao, J. M., Casas-Méndez, B., Fernández, J. R., 2014. Weighted multiple majority games with unions: Generating functions and applications to the European Union. *European Journal of Operational Research* 198, 530–544.
- Aumann, R. J., Shapley, L. S., 1974. *Values of Non-Atomic Games*. Princeton University Press.
- Béal, S., Casajus, A., Huettner, F., Rémila, E., Solal, P., 2014. Solidarity within a fixed community. *Economics Letters* 125, 440–443.
- Béal, S., Ferrières, S., Rémila, E., Solal, P., 2016. Axiomatic characterizations under players nullification. *Mathematical Social Sciences* 80, 47–57.
- Dehez, P., Tellone, D., 2013. Data games: Sharing public goods with exclusion. *Journal of Public Economic Theory* 15, 654–673.
- Derks, J., Haller, H. H., 1999. Null players out? Linear values for games with variable supports. *International Game Theory Review* 1, 301–314.
- Gómez-Rúa, M., Vidal-Puga, J., 2010. The axiomatic approach to three values in games with coalition structure. *European Journal of Operational Research* 207, 795–806.
- Graham, D. A., Marshall, R. C., Richard, J.-F., 1990. Differential payments within a bidder coalition and the Shapley value. *American Economic Review* 80, 493–510.
- Haeringer, G., 2006. A new weight scheme for the Shapley value. *Mathematical Social Sciences* 52, 88–98.

- Hammer, P. L., Peled, U. N., Sorensen, S., 1977. Pseudo-boolean functions and game theory. I. Core elements and Shapley value. *Cahiers du CERO* 19, 159–176.
- Harsanyi, J. C., 1959. A bargaining model for cooperative n -person games. In: Tucker, A. W., Luce, R. D. (Eds.), *Contributions to the Theory of Games IV*. Princeton University Press, pp. 325–355.
- Hart, S., Mas-Colell, A., 1989. Potential, value, and consistency. *Econometrica* 57, 589–614.
- Herings, P. J.-J., van der Laan, G., Talman, A. J. J., 2008. The average tree solution for cycle-free graph games. *Games and Economic Behavior* 62, 77–92.
- Huettnner, F., 2015. A proportional value for cooperative games with a coalition structure. *Theory and Decision* 78, 273–287.
- Kalai, E., Samet, D., 1987. On weighted Shapley values. *International Journal of Game Theory* 16, 205–222.
- Littlechild, S. C., Owen, G., 1973. A simple expression for the Shapley value in a special case. *Management Science* 20, 370–372.
- Maschler, M., Owen, G., 1989. The consistent Shapley value for hyperplane games. *International Journal of Game Theory* 18, 389–407.
- Megiddo, N., 1974. On the nonmonotonicity of the bargaining set, the kernel, and the Nucleolus of a game. *SIAM Journal on Applied Mathematics* 27, 355–358.
- Monderer, D., Samet, D., Shapley, L. S., 1992. Weighted values and the core. *International Journal of Game Theory* 21, 27–39.
- Moretti, S., Patrone, F., 2008. Transversality of the Shapley value. *TOP* 16, 1–41.
- Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory* 16, 161–186.
- Myerson, R. B., 1977. Graphs and cooperation in games. *Mathematics of Operations Research* 2, 225–229.
- Myerson, R. B., 1980. Conference structures and fair allocation rules. *International Journal of Game Theory* 9, 169–182.

- Naor, M., 2005. On fairness in the carpool problem. *Journal of Algorithms* 55, 93–98.
- Neyman, A., 1989. Uniqueness of the Shapley value. *Games and Economic Behavior* 1, 116–118.
- Ortmann, K., 2000. The proportional value for positive cooperative games. *Mathematical Methods of Operations Research* 51, 235–248.
- Shapley, L. S., 1953a. Additive and non-additive set functions. Ph.D. thesis, Princeton University.
- Shapley, L. S., 1953b. A value for n -person games. In: *Contribution to the Theory of Games* vol. II (H.W. Kuhn and A.W. Tucker eds). *Annals of Mathematics Studies* 28. Princeton University Press, Princeton.
- Shapley, L. S., 1971. Cores of convex games. *International Journal of Game Theory* 1, 11–26.
- Shapley, L. S., Shubik, M., 1967. Ownership and the production function. *Quarterly Journal of Economics* 81, 88–111.
- Shubik, M., 1962. Incentives, decentralized control, the assignment of joint costs and internal pricing. *Management Science* 8, 325–343.
- Stanley, R. P., 1986. *Enumerative Combinatorics*. Vol. 1. Wadsworth, Monterey, CA.
- Tijs, S. H., Driessen, T. S. H., 1986. Extensions of solution concepts by means of multiplicative ε -tax games. *Mathematical Social Sciences* 12, 9–20.
- van den Brink, R., Levínský, R., Zelený, M., 2015. On proper Shapley values for monotone TU-games. *International Journal of Game Theory* 44, 449–471.
- van den Brink, R., van der Laan, G., Pruzhansky, V., 2011. Harsanyi power solutions for graph-restricted games. *International Journal of Game Theory* 40, 87–110.
- Young, H. P., 1985. Monotonic solutions of cooperative games. *International Journal of Game Theory* 14, 65–72.

Chapter 4

An axiomatization of the iterated h -index and applications to sport rankings

Nous étudions, dans ce chapitre, une variante de l'index de Hirsch, appelée le h -index itéré et introduite par [García-Pérez \(2009\)](#) afin d'évaluer la productivité des chercheurs. Cet index se présente sous la forme d'un vecteur de h -index et pallie à un des inconvénients du h -index en permettant de classer lexicographiquement les chercheurs ayant le même h -index. Deux types de résultats sont présentés. Premièrement, nous fournissons une caractérisation axiomatique de cet index qui s'appuie sur un nouvel axiome de cohérence et sur une extension à un cadre plus riche d'axiomes existant dans la littérature. Deuxièmement, pour trois sports opposant deux équipes ou deux joueurs (tennis, basketball et football), nous utilisons le h -index et le h -index itéré afin d'obtenir un classement alternatif aux classements officiels. Ces applications révèlent clairement que le h -index itéré est bien plus approprié que le h -index classique.

Authors: Sylvain Béal, Sylvain Ferrières, Éric Rémila, Philippe Solal

Status: Submitted

Abstract:

A variant of the h -index introduced in [García-Pérez \(2009\)](#), called the iterated h -index, is studied to evaluate the productivity of scholars. It consists of successive applications of the h -index so as to obtain a vector of h -indices. In particular, the iterated h -index fixes a drawback of the h -index since it allows for (lexicographic) comparisons of scholars with the same h -index. Two types of results are presented. Firstly, we provide an axiomatic characterization of the iterated h -index, which rests on a new axiom of consistency and extensions of axioms in the literature to a richer framework. Secondly, we apply the h -index and iterated h -index to offer alternative sport rankings in tennis, football and basketball. These applications clearly demonstrate that the iterated h -index is much more appropriate than the classical h -index.

Keywords: h -index, iterated h -index, consistency axiom, sport rankings.

4.1 Introduction

The h -index ([Hirsch, 2005](#)) evaluates the individual performance of scholars based on the publications and their citations. It is equal to the integer h if h of his or her publications have at least h citations each, and his or her other publications have at most h citations each. [Hirsch \(2005\)](#) shows that the h -index is very suitable to measure the scientific production of theoretical physicists. Ever since, the h -index has been very popular and is nowadays widely used in numerous academic domains.

Nevertheless, the h -index suffers from some drawbacks inherent to its simplicity. For instance, a scholar with few fundamental publications possessing each a huge number of citations has a small h -index. Many variants of the h -index have been proposed to cope with these difficulties (see for instance [Bornmann et al., 2011](#), among others). Another drawback is that many scholars typically end up with equal small h -index, which means that the h -index cannot discriminate among them. This article considers a richer framework than the one usually considered in the literature and studies a variant of the h -index introduced in [García-Pérez \(2009\)](#), called the iterated h -index, to deal with this last problem. Our framework is richer in that an index assigns to each publication/citation vector a vector of integers with the following lexicographic interpretation. If a first index vector contains at least as many components as a second index vector, and if these components are at least as large as in the second index vector, then the scholar associated with the first index vector is considered as at least as productive as the scholar associated with the second index vector. We think that too much information is perhaps lost when computing one-dimensional indices. In this article, the (possibly) multi-dimensional indices can be seen as a trade-off between the original data (the publication vector) and a one-dimensional index. The iterated h -index belongs to this category: it contains possibly many components (dimensions), each of which resulting from the application of the classical h -index to a specific subset of the publication vector. In particular, the iterated h -index has at most as many components as the number of publications of the studied scholar. Our iterated h -index only slightly differs from the so-called multidimensional h -index in [García-Pérez \(2009\)](#) with respect to the treatment of non-cited publications. We obtain two types of results.

Firstly, we provide an axiomatic characterization of the iterated h -index by means of five axioms that are either new or adapted from axioms invoked in several

characterizations of the h -index in the simpler classical framework. The recent and growing literature on the axiomatic characterizations of the h -index has been initiated in [Woeginger \(2008a\)](#). The first axiom imposes that the index has a unique component equal to one in the benchmark case where the scholar has a unique cited publication with a unique citation. The second axiom states that the index should be multiplied by an integer c if first, the number of citations of each publication is multiplied by c and second, the resulting publication vector is replicated c times (adapted from [Quesada, 2011b](#)). The third axiom requires that the index should not vary if the number of citations of only the “best” publications increases. In the classical framework, similar axioms are called Independence of irrelevant citations and Head-independence in [Quesada \(2011b\)](#) and [Kongo \(2014\)](#), respectively. The fourth axiom states that the first components of the index should not be affected if publications with a small number of citations are added. The fifth axiom imposes that if the “best” publications are removed, then the resulting index should be obtained from the original one by removing its “best” components. In other words, if two scholars a and b differ only with respect to the “best” publications in the sense that the publication vector of scholar a is obtained from the publication vector of b by deleting b ’s “best” publications, then a ’s index should be obtained from b ’s index by deleting its “best” components. This axiom of consistency is new and is key to distinguish the iterated h -index from the h -index. Beyond the above-mentioned articles, other characterizations of the h -index are contained in [Woeginger \(2008b\)](#), [Quesada \(2010, 2011a\)](#), [Hwang \(2013\)](#), [Miroiu \(2013\)](#) and [Bouyssou and Marchand \(2014\)](#), where the latter article compares various indices from an axiomatic perspective. Other axiomatic approaches to construct index of scientific performance are developed in [Palacios-Huerta and Volij \(2004, 2014\)](#), [Chambers and Miller \(2014\)](#), [Bouyssou and Marchand \(2016\)](#) and [Perry and Reny \(2016\)](#), among others. For completeness, let us mention that [García-Pérez \(2009\)](#) does not provide axiomatic foundations of his multidimensional h -index. Beyond introducing the multi-dimensional h -index, [García-Pérez \(2009\)](#) presents some of its properties, and calibrates the productivity of professors of Methodology of the Behavioral Sciences in Spain.

Secondly, we apply both the h -index and the iterated h -index to sport rankings. More specifically, our approach is adapted to sports with duels such as tennis, football and basketball. For such a sport, the list of publications of a scholar is replaced by the list of matches won by a player or a team, while the number of citations

of each publication is replaced by the number of match won by each player/team defeated by the studied player/team. Based on the 2106 European football leagues and NBA regular seasons, we clearly underline that the h -index has a limited ranking power in that too many players/teams end up with the same h -index, even if they have very different seasonal records. To the contrary, this is not much less the case with the iterated h -index. We also point out that the iterated h -index can be used as a good proxy for NBA ranking, and provides new insights for ATP tennis ranking. For the case of European football leagues, where typically several teams are close to each other in the ranking, the use of the ih -index can lead to substantial changes. As an example, in the 2015 French league, Rennes would move from position 9 to position 15, losing approximately 3 millions euros in the distribution of the TV rights associated to the current season's performance. We also discuss the impact of the competition structures of these sports on the iterated h -index.

The rest of the article is organized as follows. Section 4.2 provides definitions and notation. Section 4.3 introduces and motivates our axioms, and states and proves the axiomatic characterization of the iterated h -index. Section 4.4 presents the application to sport rankings. Section 4.5 concludes.

4.2 Index and iterated index

4.2.1 A richer class of indices

A scholar with some publications is formally described by a vector $x = (x_1, \dots, x_{n_x})$ with nonnegative integer components $x_1 \geq x_2 \geq \dots \geq x_{n_x}$; the k th component x_k of this vector states the total number of citations to this scholar's k th-most important publication. Let X denote the set of all finite vectors x , including the empty vector. For any $x \in X$, n_x^0 denotes the number of cited publications, i.e. $n_x^0 = \max\{k = 1, \dots, n_x : x_k > 0\}$. We say that a vector $x = (x_1, \dots, x_{n_x})$ is dominated by a vector $y = (y_1, \dots, y_{n_y})$, if $n_x \leq n_y$ holds and if $x_k \leq y_k$ for $k = 1, \dots, n_x$; we will write $x \leq y$ to denote this situation.

An (generalized) index is a function $f : X \longrightarrow X$ that assigns to each $x \in X$ a vector $f(x) = (f_1(x), \dots, f_{q_x}(x))$ such that

- if $x = \emptyset$ or $x = (0, \dots, 0)$, then $f(x) = \emptyset$;

- if $x \leq y$, then $f(x) \leq f(y)$.

The first item requires that the index is empty (i.e. has zero coordinate or equivalently $q_x = 0$) for each vector without any citation. The interpretation that we propose for the index is based on lexicographic comparisons. A scholar x is considered as at most as productive as a scholar y if $f(x)$ is lexicographically dominated by $f(y)$.¹ For the rest of the article, for any index f and any vector x , keep in mind that n_x and q_x stands for the number of components in x and $f(x)$, respectively.

For an index f on X , $x \in X$ and $c = 1, \dots, q_x$, let $s(f, x, c) = \sum_{k=1}^c f_k(x)$, and set $s(f, x, 0) = 0$ by convention. Abusing notation, if $x = (x_1, \dots, x_{n_x})$, we shall sometimes write $f(x_1, \dots, x_{n_x})$ instead of $f((x_1, \dots, x_{n_x}))$. Finally, let $X^1 \subseteq X$ be the (sub)class of vectors x such that $x_{n_x} \geq n_x$. In this generalized setup, we restate the h -index and introduce an iterated version of it.

The h -index assigns to each publication vector an integer h if h publications have at least h citations each, and if the other publications have at most h publications each. Below is the definition of the h -index adapted to our richer framework.

Formally, the **h -index** (Hirsch, 2005) is the index h on X which assigns to each $x \in X$ the vector $h(x) = (h_1(x))$ such that

$$h_1(x) = \max\{k = 1, \dots, n_x : x_k \geq k\} \quad (4.1)$$

if $x_1 > 0$ and $h_1(x) = \emptyset$ otherwise.

The iterated h -index consists of several successive applications of the h -index. More specifically, its first component is obtained by a first classical application of the h -index. If this h -index is equal to c , then the most c -th cited publications are removed, and the h -index is applied another time to the resulting smaller vector. This yields the second component of the iterated h -index. This step is repeated until all cited publications have been treated. As such, the iterated h -index permits to discriminate among scholars with the same h -index.

Formally, the **iterated h -index** is the index ih on X which assigns to each

¹Other interpretations are discussed in section 4.3.3.

$x \in X$ the vector $ih(x) = (ih_1(x), \dots, ih_{q_x}(x))$ such that for all $k = 1, \dots, q_x$,

$$ih_k(x) = \max\{c = 1, \dots, n_x - s(ih, x, k-1) : x_{s(ih, x, k-1)+c} \geq c\}$$

and $ih(x) = \emptyset$ if x is either empty or $x_1 = 0$.

By definition, $ih_1(x) = h_1(x)$, and $ih_1(x) \geq \dots \geq ih_{q_x}(x)$. Note also that $h(x) = ih(x) = (n_x)$ for all $x \in X^1$. Furthermore, it is easy to check that $\sum_{k=1}^{q_x} ih_k(x) = n_x^0$, i.e. the sum of the components' value of the iterated h -index add up to the number of cited publications. The iterated h -index is the same as the multidimensional h -index in [García-Pérez \(2009\)](#), except that we associate with empty vectors or non-cited publications an empty component whereas [García-Pérez \(2009\)](#) uses a zero component.

As an example, pick $x = (9, 9, 7, 6, 6, 5, 4, 4, 2, 1, 1, 0)$. Then one has $h(x) = (5)$ and $ih(x) = (5, 3, 1, 1, 1)$. These computations are even easier to grasp by drawing the picture represented in [Figure 4.1](#).

As mentioned in the introduction, our aim is to distinguish among scholars characterized by the same h -index. This is a reason why we use lexicographic comparisons. Hence, if $x = (9, 9, 7, 6, 6, 5, 4, 4, 2, 1, 1, 0)$ as before and if $y = (6, 6, 6, 6, 6, 6)$, so that $ih(y) = (6)$, then we consider that scholar x is less productive than scholar y .

4.2.2 Operations on X

For any vectors $x \in X$ and $y \in \mathbb{N}_*^{n_x}$, define the **addition of x and y** as the vector $(x + y)$ of dimension n_x such that $(x + y)_k = x_k + y_k$ for each $k = 1, \dots, n_x$.

For any $x \in X$ and $c \in \mathbb{N}$, the **c -expansion of x** is the vector denoted by $(c \otimes x) \in X$ of dimension cn_x is defined, for all $k = 1, \dots, cn_x$ as $(c \otimes x)_k = cx_{\lceil k/c \rceil}$, where for each real number $a \in \mathbb{R}_+$, $\lceil a \rceil$ is the smallest integer greater than or equal to a . In words, the number of each citation in x is multiplied by c and then, the resulting publications are copied c times. Also, for any $x \in X$ and $c \in \mathbb{N}$, the **c -multiplication of x** is the vector $cx \in X$ is given by (cx_1, \dots, cx_{n_x}) . As an example, if $x = (4, 4, 3, 1)$ and $c = 3$, then $(c \otimes x) = (12, 12, 12, 12, 12, 12, 9, 9, 9, 3, 3, 3)$ and $cx = (12, 12, 9, 3)$.

For all $x \in X$ and $c \in \mathbb{N}$, define $d(x, c) = \arg \min_{k=1, \dots, n_x} \{x_k : x_k < c\}$ if $c > x_{n_x}$ and $d(x, c) = n_x + 1$ if $c \leq x_{n_x}$ as the lowest position in x such that the associated

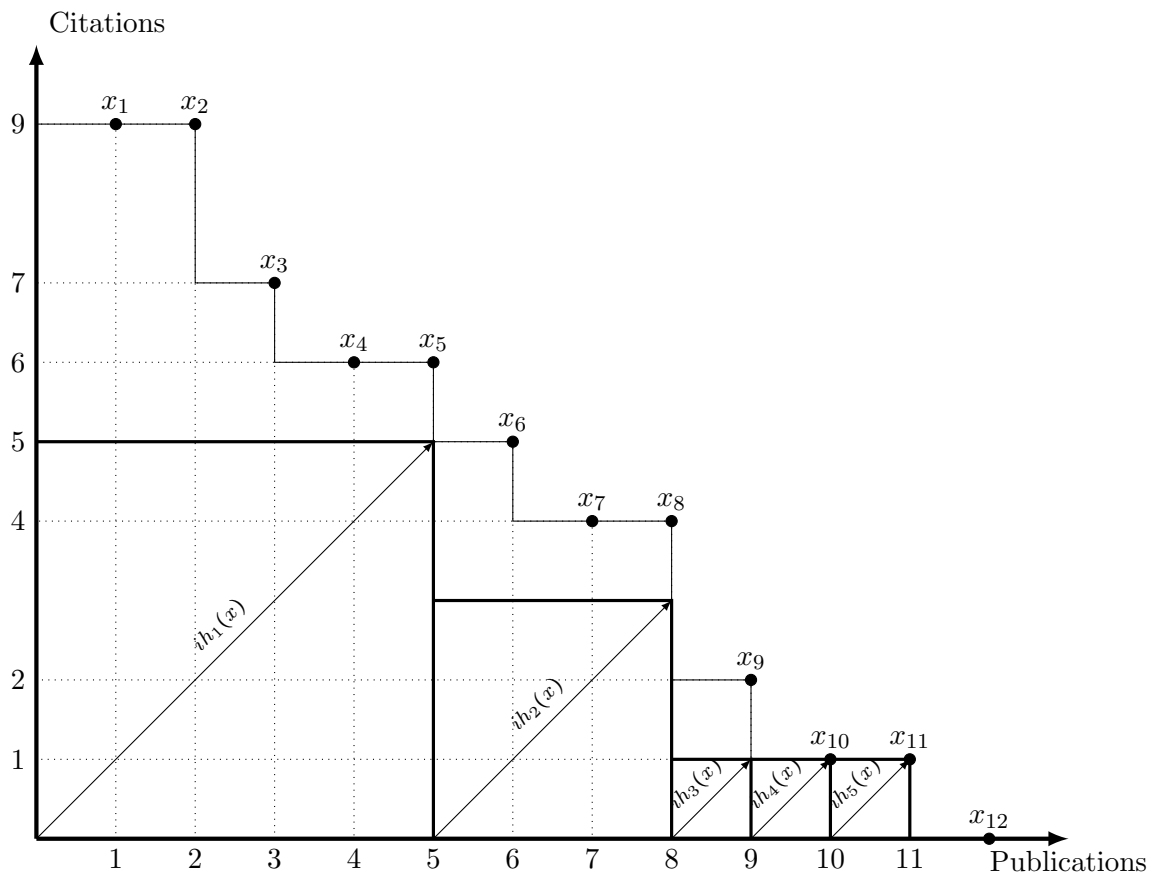


Figure 4.1: Graphical representation

publication as strictly less than c citations if such a position exists and $x_{n_x} + 1$ otherwise. Furthermore, for all $x \in X$ and $c \in \mathbb{N}$ define the **union of x with a c -cited publication** as the vector $x \cup (c)$ obtained from x by adding a publication with c citations in position $d(x, c)$ (each less-cited publication being moved from its original position to the immediately next one). Formally:

- $(x \cup (c))_k = x_k$ if $x_k \geq c$;
- $(x \cup (c))_k = x_{k-1}$ if $x_{k-1} < c$;
- $(x \cup (c))_{d(x,c)} = c$.

As an example, if $x = (6, 5, 5, 4, 3, 1, 1)$ and $c = 4$, then $d(x, c) = 5$ (since the publication with 3 citations is in position 5) and $x \cup (c) = (6, 5, 5, 4, \mathbf{4}, 3, 1, 1)$ where the newly added publication is highlighted in bold.

For all $x \in X$ and $k = 1, \dots, n_x$, define the **vector x without its k -th most cited publication** as $x \setminus (x_k) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n_x})$.

4.3 Axiomatic study

We begin this section by listing the axioms that we invoke. References to versions of the axiom already existing in the literature on the h -index are given in the introduction of the article and are not repeated here. Then we demonstrate the main characterization as well as an instructive preliminary result.

4.3.1 Axioms

This first axiom is new and provides a benchmark or normalization. If a researcher has a unique cited publication (and thus possibly many publications without any citation), and if this publication has received a unique citation, then the index has a unique component equal to 1.

One citation case (OC) If $\sum_{k=1}^{n_x} x_k = 1$, then $f(x) = (1)$.

The second axiom says that adding citations to the $f_1(x)$ -th most cited publications has no impact on the index. The associated publications can be considered as the best of the studied scholar, and the axiom means that extra citations for these publications does not improve the scholar's productivity, *ceteris paribus*. In this sense, the added citations can be considered as superfluous.

Independence of superfluous citations (ISC) For all $x \in X$ and $y \in \mathbb{N}^{n_x}$ such that $y_k = 0$ whenever $k > f_1(x)$, if $(x + y) \in X$ then $f(x + y) = f(x)$.

The third axiom states that multiplying by c the number of citations and then by c the number of publications (as in a c -expansion) amounts to multiply by c the index. In other words, a change in scale of the scholar's production vector leads to the same change in scale for each of his/her index's component.

Homogeneity (H) For all $x \in X$ and all $c \in \mathbb{N}$, $f(c \otimes x) = cf(x)$.

The fourth axiom states that adding publications with at most $f_k(x)$ citations has no impact on the first k components of the index. In this sense, such ("weak") publications can be considered as irrelevant for these ("better") components. In order to state this axiom, for any index f on X and any $x \in X$, we adopt the convention $f_{q_x+1}(x) = 0$.

Independence of irrelevant publications (IIP) For all $x \in X$, all $k = 1, \dots, q_x + 1$ and all $c \in \mathbb{N}$ such that $c \leq f_k(x)$, it holds that $f_j(x) = f_j(x \cup (c))$ for each $j = 1, \dots, k$.

The last axiom involves the most $s(f, x, c)$ -th cited publications. It states that these publications are removed (and are "rewarded" according to the first c components of the index in a sense), then the index of the new situation is the original index deprived of its first c components. In an other sense, the axiom means that deleting the best publications does not change the last components of the index.²

²A more general version of this axiom can be stated by removing the publications "associated with" any set of components.

Consistency (C) For all $x \in X$ and $c = 1, \dots, q_x$, we have $f(x \setminus (x_1, \dots, x_{s(f,x,c)})) = f(x) \setminus (f_1(x), \dots, f_c(x))$.

4.3.2 Results

We start by proving a preliminary result on the class X^1 , which states that axioms **OC**, **ISC** and **H** already characterize the classical h -index for the particular publication vectors in X^1 . Since the ih -index coincides with the h -index on X^1 , this result also characterizes the ih -index on this class.

Lemma 4.1. *An index f on X^1 satisfies **OC**, **ISC** and **H** if and only if $f = h$.*

Before proving Lemma 4.1, note that **OC**, **ISC** and **H** are well-defined on X^1 . More specifically, among the vectors of the form $x = (1, 0, \dots, 0)$ that can be considered in **OC**, only $x = (1)$ belongs to X^1 . Furthermore, for any $x \in X^1$ and $y \in \mathbb{N}^{n_x}$, note also that the vector $(x + y)$ is in X^1 if and only if $(x + y)$ is in X . Similarly, for any $x \in X^1$ and any $c \in \mathbb{N}$, $(c \otimes x) \in X^1$ as well.

Proof. It is clear that h satisfies the three axioms on X^1 , and that $h(x) = (n_x)$ for all $x \in X^1$. Conversely, consider any index f on X^1 satisfying the three axioms. Pick any $x \in X^1$, so that it must be that $x_1 > 0$. Since $x_{n_x} \geq n_x$, x can be expressed as $x = (z + y)$, with $z \in X^1$ and $y \in \mathbb{N}^{n_x}$ such that $z = (n_x, \dots, n_x)$ and $y = (x_1 - n_x, \dots, x_{n_x} - n_x)$. It holds that $z = (n_x \otimes (1))$, so that **H** implies that $f(z) = f(n_x \otimes (1)) \stackrel{\mathbf{H}}{=} n_x f(1)$. Moreover, **OC** yields that $f(1) \stackrel{\mathbf{OC}}{=} (1)$. Thus, $f(z) = n_x(1) = (n_x)$. In particular, we have $f_1(z) = n_x$. Coming back to y , since y has n_x coordinates, **ISC** can be applied to z and y : $f(x) = f(z + y) \stackrel{\mathbf{ISC}}{=} f(z) = (n_x)$. Conclude that $f(x) = h(x)$. ■

Proposition 4.1 below relies on Lemma 4.1 and add axioms **IIP** and **C** in order to characterize the ih -index on the full domain of publication vectors.

Proposition 4.1. *An index f on X satisfies **OC**, **ISC**, **H**, **IIP** and **C** if and only if $f = ih$.*

Proof. It is easy to check that ih satisfies the five axioms on X . Conversely, let f be any index satisfying the five axioms on X . To show that f is uniquely determined.

So let $x \in X$. Since $f(x) = \emptyset$ if x is either empty or $x_1 = 0$ by definition of an index f , we shall only consider vectors x with some cited publications. For each $k = 1, \dots, q_x$, denote by $x^{(k)}$ the sub-vector of x containing the publications in x whose position is between $s(ih, x, k-1) + 1$ and $s(ih, x, k)$, that is, $x^{(k)} = (x_{s(ih, x, k-1)+1}, \dots, x_{s(ih, x, k)})$. So $x^{(1)}$ contains the $ih_1(x)$ -th most cited publications, $x^{(2)}$ the next most $ih_2(x)$ -th cited publications and so on until all cited publications have been taken into account. For each $k = 1, \dots, q_x$, by definitions of ih and $x^{(k)}$, it holds that $x^{(k)} \in X^1$ since $x_{n_{x^{(k)}}}^{(k)} \geq ih_k(x) = s(ih, x, k) - s(ih, x, k-1) = n_{x^{(k)}}$. In particular, if $x \in X^1$, then $x = x^{(1)}$. Furthermore, it is easy to check that $ih_1(x^{(k)}) = ih_k(x)$ for each $k = 1, \dots, q_x$. Thus, by Lemma 4.1, we have

$$f(x^{(k)}) = (f_1(x^{(k)})) = (ih_1(x^{(k)})) = (ih_k(x)) \quad (4.2)$$

for each $k = 1, \dots, q_x$. For the rest of the proof, we demonstrate that $f_k(x)$ coincides with $ih_k(x)$ by induction on k .

INITIALIZATION. For $k = 1$, from the previous arguments $ih_1(x) = ih_1(x^{(1)}) = f_1(x^{(1)})$, and $ih_1(x) \geq s(ih, x, 1)$. Furthermore, for any $j \geq s(ih, x, 1)$, $x_j \in x^{(k)}$ for some $k = 2, \dots, q_x$, and thus $x_j \leq s(ih, x, 1)$. This means that **IIP** can be used to obtain $f_1(x^{(1)}) \stackrel{\text{IIP}}{=} f_1(x^{(1)} \cup x^{(q_x)})$. Thus, repeated applications of **IIP** yield

$$ih_1(x) = ih_1(x^{(1)}) = f_1(x^{(1)}) \stackrel{\text{IIP}}{=} f_1(x^{(1)} \cup \dots \cup x^{(q_x)}) = f_1(x)$$

which means that $f_1(x) = ih_1(x)$ as desired.

INDUCTION HYPOTHESIS. Assume that $f_k(x) = ih_k(x)$ for each $k < q$, $q = 2, \dots, q_x$.

INDUCTION STEP. Consider the component $f_q(x)$ of $f(x)$. Since each component $f_k(x)$, $k = 1, \dots, q-1$, is known and coincides with $ih_k(x)$, $k = 1, \dots, q-1$, by the induction hypothesis, an application of **C** yields that

$$\begin{aligned} f(x^{(q)} \cup \dots \cup x^{(q_x)}) &= f(x \setminus (x^{(1)} \cup \dots \cup x^{(q-1)})) \\ &\stackrel{\text{C}}{=} f(x) \setminus (f_1(x), \dots, f_{q-1}(x)) \\ &= (f_q(x), \dots, f_{q_x}(x)). \end{aligned}$$

In particular, this means that $f_q(x^{(q)} \cup \dots \cup x^{(q_x)}) = f_q(x)$. Moreover, similarly as in

the initialization, by **IIP**, we can write that

$$f_q(x^{(q)} \cup \dots \cup x^{(q_x)}) \stackrel{\mathbf{IIP}}{=} f_q(x^{(q)}) = ih_q(x),$$

where the last equality is the consequence of Lemma 4.1 pointed out in (4.2). Thus $f_q(x) = ih_q(x)$ for all $q = 1, \dots, q_x$. This means that the most $s(ih(x, q_x) = n_x^0$ -th cited publications have been treated. By **C**, we have

$$f(x \setminus (x_1, \dots, x_{n_x^0})) \stackrel{\mathbf{C}}{=} f(x) \setminus (f_1(x), \dots, f_{q_x}(x))$$

Since $(x \setminus (x_1, \dots, x_{n_x^0}))$ is either empty or of the form $(0, \dots, 0)$, The left-hand side is empty by definition of an index. As a consequence, the right-hand side is empty too, proving that f cannot have more nonempty coordinates. This completes the proof that $f = ih$. \blacksquare

It is worth noting that Proposition 4.1 provides an alternative formulation of the iterated h -index: for any $x \in X$, and $k = 1 \dots, q_x$, it is given by $ih_k(x) = h(x^{(k)})$. The proof that the axioms in Proposition 4.1 are logically independent is made by exhibiting the following index on X :

- The h -index on X satisfies **OC**, **ISC**, **H**, **IIP** but violates **C**.
- The index f on X such that for each $x \in X$, $f(x) = \emptyset$ satisfies **ISC**, **H**, **IIP**, **C** but violates **OC**.
- The index f on X such that for each $x \in X$, $f(x) = (1)$ if $x_1 > 0$ and $f(x) = \emptyset$ otherwise satisfies **OC**, **ISC**, **IIP**, **C** but violates **H**.
- The index f on X such that for each $x \in X$, $f(x) = (n_x^0)$ if $n_x^0 \neq 0$ and $f(x) = \emptyset$ otherwise satisfies **OC**, **ISC**, **H**, **C** but violates **IIP**.
- The index f on X such that for each $x \in X$, $f(x) = (\min_{k=1, \dots, n_x} \{n_k : x_k > 0\})$ if $x_1 > 0$ and $f(x) = \emptyset$ otherwise satisfies **OC**, **H**, **IIP**, **C** but violates **ISC**.

As a final remark, we can suggest a characterization of the h -index in our framework of possibly multidimensional indices. As pointed out in the preceding paragraph the h -index satisfies **OC**, **ISC**, **H** and **IIP**. The combination of these

four axioms is not sufficient to characterize the h index by Proposition 4.1. A characterization can be obtained by strengthening axiom **IIP** as follows.

Strong independence of irrelevant publications (SIIP) For all $x \in X$, all $k = 1, \dots, q_x + 1$ and all $c \in \mathbb{N}$ such that $c \leq f_1(x)$, it holds that $f(x) = f(x \cup (c))$.

This new axiom imposes that an index is invariant to adding a new publication with at most as citations as the first component of the index. It shares some similarities with axiom C14 – Square rightwards in Bouyssou and Marchand (2014). Combining **OC**, **ISC** and **H** with **SIIP** yields a characterization of the h -index given by (4.1). The proof is similar to those of Proposition 4.1 and is omitted.

4.3.3 Discussion

Alternative interpretations Until now, we have adopted a lexicographic interpretation in order to compare scholars by means of their respective ih -index. Many other criteria are conceivable. As an example, scholars can also be compared via the Lorenz dominance (see Sen, 1973, for an introduction to this literature). Consider two scholars x and y , and their ih -index $ih(x) = (ih_1(x), \dots, ih_{q_x}(x))$ and $ih(y) = (ih_1(y), \dots, ih_{q_y}(y))$, respectively. According to the Lorenz dominance, scholar x is said to be as productive as scholar y if for all $k \in \{1, \dots, \max\{q_x, q_y\}\}$, it holds that

$$\sum_{i=1}^k ih_i(x) \geq \sum_{i=1}^k ih_i(y).$$

Contrary to the lexicographic interpretation, it is obvious that the Lorenz domination does not yield a total order on the set of scholars, as pointed out in the next example. Suppose that $x = (12, 9, 9, 7, 7, 6, 3, 2)$ and $y = (9, 8, 7, 7, 7, 6, 5, 4)$, so that $ih(x) = (6, 2)$ and $ih(y) = (5, 4)$. Scholars x and y cannot be compared by using the Lorenz dominance since $ih_1(x) > ih_1(y)$ but $ih_1(x) + ih_2(x) < ih_1(y) + ih_2(y)$.

Another variant in the same spirit The ih -index improves upon the h -index by processing the information contained in the tail of the publication record, i.e. by gratifying the citations of the least-cited publications. Another variant of the h -index can be constructed by considering more finely the head of the distribution

instead of its tail. More specifically, the h -index potentially excludes some of the citations of the most-cited publications. Similarly to the ih -index, it is possible to further apply the h -index to these “remaining” citations, leading to a new multidimensional index, exactly as what the ih -index does for the “remaining” publications. In order to be clear, let us come back to the example depicted in Figure 4.1. The h -index singles out the five best publications. However, the first two publications have each 4 more citations than necessary to attain this result. Similarly, the three other concerned publications have 2, 1 and 1 more citations than necessary, respectively. Applying iteratively the above-mentioned principle, we obtain the multidimensional index $(5, 2, 2)$, where the first 2 indicates that within the five best publications, two have at least two “remaining” citations each, and the three other have at most two “remaining” citations each. The last 2 has a similar interpretation except that it deals only with the best two publications, and that the number of “remaining” citations is calculated after removing $5 + 2$ citations. [García-Pérez \(2012\)](#) even considers the possibility to combine extensions of the h -index in the tail and head areas simultaneously.

4.4 Alternative sport rankings

In this section, we consider several sport competitions in which the ih -index can be calculated. ATP tennis, NBA basketball and European national football leagues are investigated. The first objective is to determine whether the ih -index provides relevant alternative rankings to the official rankings (ATP ranking for tennis, winning percentage for NBA, and total points by receiving three points for a win and one point for a draw for football leagues). The second objective is to discuss how the various competition formats influence the ih -index.

4.4.1 Tennis

For the computation of the ih -index, we have extracted data from the official website of the ATP (<http://www.atpworldtour.com/>). International tennis competition is mostly based on five types of events: the four grand slam, the ATP world tour, which includes the other most prestigious tournaments, the ATP challenger tour and the ITF circuit, composed of less prestigious tournaments, and the Davis

cup, a team competition. In what follows, we only take into account grand slam and the ATP world tour matches, including qualification matches. Each professional player is associated with a vector in which each integer is the number of wins achieved a player he has defeated at grand slams and ATP world tour tournaments.

As an example, Mikhail Youzhny (ranked 127th at the 2015 ATP year-end ranking) is associated with vector (41, 38, 23, 23, 22, 20, 19, 18, 18, 17, 14, 13, 10, 9, 0). This means that for the 2015 ATP season, Youzhny has won 15 grand slam and the ATP world tour matches, including qualifications. Among these wins, Gilles Simon is the defeated player with most wins (41), Viktor Troicki is the defeated player with the second most number of wins (38), and so on. The zero at the end of the vector corresponds to Youzhny's win against Yassine Idmbarek, a low-ranked player who had no win at grand slam and the ATP world tour level in 2015. The ih -index of Mikhail Youzhny is thus (12, 2).

Table 4.1 summarizes, for the 2015 season, the ih -index (and the corresponding ranking), the ATP year-end ranking, the total number of ATP points, and the differences in these rankings for the top 50 players (according to the ih -index). If two players have the same total number of points or the same ih -index, ties shall be broken by using the the most total points from the grand slams as used in the official ATP ranking.

Table 4.1 reveals the following facts. The two ranking systems (ATP and ih -index) agree on the best 6 players. Moreover, 9 of the top 10 ATP players also belong to the top 10 ih -index players. Jo-Wilfried Tsonga (ATP-10, ih -index-17) is the only exception, because an injury prevented him from playing as many tournaments as the other top 10 players. Regarding the ih -index top 50, 47 players also belong to the ATP top 50, with some notable differences explained below. The major difference between the two ranking systems have four main sources. Firstly, as for Tsonga, some players have played less tournaments than the average, even if they enjoyed good performances. Beyond Tsonga, this is the case for Cilic, among others. Secondly, some players have played at the ATP challenger tour level, or even at the ITF future tour level. Among the best players, Benoit Paire is an example. He started the 2015 season with a low ranking, which forces him to play less prestigious tournament during the first tier of the season. Since we do not count such tournaments in our study, it is not surprising to observe that is ih -index ranking is lower than his ATP ranking. Similar explanations can be put

Player	<i>ih</i> -index ranking	<i>ih</i> -index	ATP ranking	ATP points	Difference
Novak Djokovic	1	(37, 26, 14, 4, 1)	1	16 585	=
Andy Murray	2	(31, 23, 6, 2, 1)	2	8 945	=
Roger Federer	3	(31, 18, 11, 1)	3	8 265	=
Stan Wawrinka	4	(30, 16, 6, 2)	4	6 865	=
Rafael Nadal	5	(28, 19, 9, 3)	5	5 230	=
Tomas Berdych	6	(27, 20, 7, 2, 1)	6	4 620	=
Key Nishikori	7	(26, 16, 7, 2)	8	4 235	▲1
John Isner	8	(25, 13, 5, 2)	11	2 495	▲3
Richard Gasquet	9	(25, 13, 4, 1)	9	2 850	=
David Ferrer	10	(24, 18, 9, 2)	7	4 305	▼3
Gilles Simon	11	(24, 12, 4)	15	2 145	▲4
Kevin Anderson	12	(23, 15, 4, 2, 2)	12	2 475	=
Roberto Bautista Agut	13	(22, 12, 4, 2)	25	1 480	▲12
Ivo Karlovic	14	(22, 12, 3)	23	1 485	▲9
Dominic Thiem	15	(22, 12, 2)	20	1 600	▲5
Gaël Monfils	16	(21, 10, 2)	24	1 485	▲8
Jo-Wilfried Tsonga	17	(21, 9, 2)	10	2 635	▼7
Milos Raonic	18	(21, 9, 2)	14	2 170	▼4
Viktor Troicki	19	(21, 8, 4)	22	1 487	▲3
Feliciano Lopez	20	(21, 8, 2, 1)	17	1 690	▼3
Guillermo Garcia-Lopez	21	(21, 8, 2)	27	1 430	▲6
Joao Sousa	22	(21, 8, 2)	33	1 191	▲11
Bernard Tomic	23	(20, 11, 3, 2, 1)	18	1 675	▼5
Steve Johnson	24	(20, 10, 6, 1)	32	1 240	▲8
David Goffin	25	(20, 10, 4)	16	1 880	▼9
Grigor Dimitrov	26	(20, 10, 2)	28	1 360	▲2
Jack Sock	27	(20, 9, 4)	26	1 465	▼1
Alexandr Dolgoplov	28	(20, 9, 2)	36	1 135	▲8
Marin Cilic	29	(19, 11, 4)	13	2 405	▼16
Simone Bolelli	30	(19, 10, 4)	58	790	▲28
Philipp Kohlschreiber	31	(19, 7, 4)	34	1 185	▲3
Fabio Fognini	32	(19, 7, 3)	21	1 515	▼11
Marcos Baghdatis	33	(19, 5, 1)	46	933	▲13
Gilles Muller	34	(18, 10, 1)	38	1 105	▲4
Nick Kyrgios	35	(18, 7)	30	1 260	▼5
Benoit Paire	36	(18, 6, 3, 1, 1)	19	1 633	▼17
Borna Coric	37	(18, 5)	44	941	▲7
Thomaz Belluci	38	(17, 8, 3, 2)	37	1 105	▼1
Adrian Mannarino	39	(17, 7, 2)	47	930	▲8
Jérémy Chardy	40	(17, 7, 1, 1)	31	1 255	▼9
Pablo Cuevas	41	(17, 7, 1)	40	1 065	▼1
Vasek Pospisil	42	(16, 7, 3, 1)	39	1 075	▼2
Andreas Seppi	43	(16, 6, 3, 1)	29	1 360	▼14
Donald Young	44	(16, 6, 1)	48	907	▲4
Martin Klizan	45	(16, 5, 4, 1, 1)	43	980	▼2
Jerzy Janowicz	46	(16, 5, 1)	57	795	▲11
Lukas Rosol	47	(16, 4, 1)	55	797	▲8
Fernando Verdasco	48	(15, 7, 2)	49	900	▲1
Tommy Robredo	49	(15, 6, 1)	42	1 000	▼7
Leonardo Mayer	50	(15, 5, 1, 1)	35	1 150	▼15

Table 4.1: Tennis season 2015.

forward for Leonardo Mayer. Thirdly, some players have been very successful in the less prestigious category of ATP world tour tournaments. Thus, they accumulated wins but not so many points. Examples of such players, having a better ih -index ranking than ATP ranking, are Dominic Thiem and Joao Sousa. Fourthly, we count qualification wins which provide only a small number of points. The ranking of some players is not good enough to enter main draws directly, so that they sometimes win many qualification matches. This is the case for Baghdatis and Bolelli, among others. They also achieve a better ih -index ranking than ATP ranking.

The tennis ranking provided by the ih -index is in line with other alternative rankings proposed in the literature, for instance by [Dahl \(2012\)](#). The ih -index is also useful to evaluate the strength of tennis players across years. In the past 2013 and 2014 ATP seasons, the players with the best ih -index were the two number one in the world: Rafael Nadal, (36, 22, 15, 5) in 2013, and Novak Djokovic, (35, 19, 9, 2) in 2014. Both players have a smaller ih -index than Novak Djokovic in 2015, which is among the best seasons ever achieved by a player on the ATP tour. In 2015, Djokovic's ih -index even surpasses Federer's ih -index (37, 24, 13, 5, 3) in his great 2006 season. Similarly, [Ruiz et al. \(2013\)](#) rely of a data envelopment analysis to assess tennis players' performances. Finally, it would be nice to determine whether the ih -index is a better predictor for the outcome of tennis matches than the ATP official ranking, which is used by [Clarke and Dyte \(2000\)](#) and [del Corrala and Prieto-Rodríguez \(2010\)](#) to predict grand slam tournaments outcomes.

4.4.2 Basketball

Data come from http://www.basketball-reference.com/leagues/NBA_2016_games.html. In the 2016 NBA regular season, each of the 30 teams plays 82 matches against each other, and the ranking among them is calculated on the basis of the winning percentage. Teams are grouped into two conferences (Eastern and Western), and the 8 top teams in each conference are qualified from a playoff tournament which determines the NBA champion. In this section, we only study the regular season, and compare the official NBA ranking with those provided by the ih -index. Statistics are contained in tables [4.2](#) and [4.3](#).

These tables call up the following comments. Firstly, for the NBA regular season, a team has qualified for the playoff via the official NBA ranking if and

Team	<i>ih</i> -index ranking	<i>ih</i> -index	Conference ranking	Winning %	Difference
Toronto Raptors	1	(35, 17, 4)	2	0.683	▲1
Cleveland Cavaliers	2	(33, 18, 6)	1	0.695	▼1
Atlanta Hawks	3	(33, 12, 3)	4	0.585	▲1
Miami Heats	4	(33, 12, 3)	3	0.585	▼1
Boston Celtics	5	(33, 11, 4)	5	0.585	=
Charlotte Hornets	6	(32, 12, 4)	6	0.585	=
Detroit Pistons	7	(32, 10, 2)	8	0.537	▲1
Indiana Pacers	8	(31, 11, 3)	7	0.549	▼1
Chicago Bulls	9	(31, 10, 1)	9	0.512	=
Washington Wizards	10	(30, 10, 1)	10	0.500	=
Orlando Magic	11	(27, 8)	11	0.427	=
Milwaukee Bucks	12	(24, 9)	12	0.402	=
New York Knicks	13	(23, 9)	13	0.390	=
Brooklyn Nets	14	(19, 2)	14	0.256	=
Philadelphia 76ers	15	(10)	15	0.122	=

Table 4.2: NBA 2016 regular season – Eastern conference.

Team	<i>ih</i> -index ranking	<i>ih</i> -index	Conference ranking	Winning %	Difference
Golden State Warriors	1	(41, 23, 9)	1	0.890	=
San Antonio Spurs	2	(37, 22, 8)	2	0.817	=
Oklahoma City Thunder	3	(33, 17, 5)	3	0.671	=
Los Angeles Clippers	4	(33, 17, 3)	4	0.646	=
Portland Trail Blazers	5	(31, 12, 1)	5	0.537	=
Memphis Grizzlies	6	(30, 10, 2)	7	0.512	▲1
Houston Rockets	7	(30, 10, 2)	8	0.500	▲1
Dallas Mavericks	8	(29, 11, 2)	6	0.512	▼2
Utah Jazz	9	(29, 10, 1)	9	0.488	=
Denver Nuggets	10	(26, 7)	11	0.402	▲1
Sacramento Kings	11	(24, 9)	10	0.402	▼1
New Orleans Pelicans	12	(24, 6)	12	0.366	=
Minnesota Timberwolves	13	(23, 6)	13	0.354	=
Phoenix Suns	14	(20, 3)	14	0.280	=
Los Angeles Lakers	15	(16, 1)	15	0.207	=

Table 4.3: NBA 2016 regular season – Western conference.

only if it has also qualified by means of the ih -index ranking. In other words, the two rankings agree on the eight first teams in both conferences, but not on their orders. Secondly, it should be noted that many teams achieve the same winning percentage (for instance 4 teams in the eastern conference), which necessitates to use tie breaking rules. In the ih -index ranking, only two teams are in that case. Thirdly, even if the lists of qualified teams are the same under the two ranking systems, there are nevertheless small changes in the rankings that can have important consequences for the playoff phase. The reason is that the position in the bracket (and so the potential advantages going with a good position, such as playing a low-ranked team and the home-court advantage) depends on the rankings in the regular season. As an example, in the eastern conference, the ranking of the top 2 teams is inverted when the ih -index replaces the official winning percentage. The consequence is that Cleveland would have lost the home-court advantage in the conference final against Toronto. The ih -index and NBA winning percentage agree on the first four teams in each conference (but not in the same order in the Eastern conference), which means that the home-court advantage would be the same with the two ranking systems in the first round of playoffs. Fourthly, there is no change in ranking for the 14 teams that did not qualify for the playoff phase. Here too, these positions are important for the NBA draft, which is the annual event during which all NBA teams can draft promising players who are eligible and wish to join the league. The reason is that these 14 worst teams are assigned the first 14 choices by a lottery in which the probability to obtain the first choice is decreasing with the team ranking. [Taylor and Trogdon \(2002\)](#) and [Price et al. \(2010\)](#) point out that teams eliminated from playoffs can strategically lose games at the end of the season in order to increase their probability to get the first draft choice, while [Lenten \(2016\)](#) shows that a team's performances increase when this perverse incentive is eliminated. [Motomura et al. \(2016\)](#) prove that building a team through the draft is not the most successful strategy. Finally, we can point out a difference with the study on tennis. Each NBA team plays a fixed number of matches. In that sense, a NBA team cannot improve its ih -index by playing more games, contrary to a tennis player who can add extra tournaments to his calendar.

Team	<i>ih</i> -index ranking	<i>ih</i> -index	League ranking	Points	Difference	2015 TV rights
Paris	1	(12, 8, 4)	1	83	=	15 714 696
Lyon	2	(12, 8, 2)	2	75	=	13 663 055
Marseille	3	(12, 7, 2)	4	69	▲1	10 323 682
Monaco	4	(12, 7, 1)	3	71	▼1	11 873 326
Saint-Etienne	5	(12, 7)	5	69	=	8 970 472
Bordeaux	6	(12, 5)	6	63	=	7 802 783
Guingamp	7	(12, 3)	10	49	▲3	4 452 497
Montpellier	8	(11, 5)	7	56	▼1	6 787 876
Lille	9	(11, 5)	8	56	▼1	5 893 011
Nice	10	(11, 2)	11	48	▲1	3 874 109
Caen	11	(11, 1)	13	46	▲2	2 924 680
Bastia	12	(11, 1)	12	47	=	3 372 112
Reims	13	(11, 1)	15	44	▲2	2 215 336
Toulouse	14	(11, 1)	17	42	▲3	1 669 686
Rennes	15	(10, 3)	9	50	▼6	5 129 102
Nantes	16	(10, 1)	14	45	▼2	2 542 725
Lorient	17	(9, 3)	16	43	▼1	1 920 685
Evian	18	(7, 4)	18	37	=	0
Lens	19	(7)	20	29	▲1	0
Metz	20	(7)	19	30	▼1	0

Table 4.4: 2015 French league.

4.4.3 Football

The main European football leagues share the same ranking system. Each team plays twice against each other team, and add 3 points to its total in case of a win, and 1 point in case of a draw. The only difference is the number of teams in the league, which is 20 for the French, Spanish, Italian and English leagues, and only 18 for the German league. Contrary to the NBA, there are no playoffs: the top-ranked team wins the championship. The league ranking determines which teams qualify for the UEFA champions league and the Europa league, and which teams are relegated to the second division league. On top of that, the ranking is also crucial for teams in order to obtain the best possible share in the TV (broadcasting) rights distribution. Table 4.4 provides an example based on the 2015 French league, where the last column indicates the share of the TV rights obtained by each teams for its current season's official ranking (the total is about 25% of the total TV rights for the French case). Data come from Wikipedia. As for tennis and basketball, we have chosen to use the same tie-breakers as for the official ranking.

Before discussing the particular case of the 2015 season, it should be noted that tie/draw results are not taken into account by the *ih*-index. As a consequence, the *ih*-index provides an incentive for teams to win that is similar to the rule giving three points for a win (instead of two) adopted by all national leagues for many years (see [Dilger and Geyer, 2009](#); [Guedes and Machado, 2002](#), for instance). The fairness of the three-point rule is sometimes disputed as underlined in [Bring and Thures-](#)

son (2011), and we think that ih -index can be considered as a relevant consensual alternative.

Table 4.4 reveals some substantial differences between the ih -index and the official league ranking. Firstly, even if the five teams qualified for the European competitions are the same, the third spot for the champions league goes to Marseille instead of Monaco if the ih -index replaces the official ranking. Note also that the three relegated teams are the same with both rankings too. Secondly, the difference in rankings for some other teams is not negligible. For instance, Rennes falls from position 9 with the official ranking to position 15 with the ih -index ranking, which would translate into a loss of money of around 2.91 million euros. Furthermore, the h -index is obviously limited here since seven teams obtain an h -index of 12, while seven other teams obtain an h -index of 11. The evident explanation is that European football leagues feature a smaller number of matches per teams than the in a NBA regular season or than the number of annual matches for the best ATP players.

4.4.4 Discussion

The three applications to sport ranking considered so far clearly indicate that the (classical) h -index is perhaps not a good tool to rank teams and players, since many of them end up with the same h -index, even if they have very different season records. To the contrary, the ih -index has several components from which teams and players with the same h -index can be distinguished. Even in sports for which the regular season contains many games, the h -index could have a limited power. For instance, in the Major league baseball, teams play around 160 games during the season. The official ranking is the winning percentage as for the NBA basketball, but the difference in winning percentage between the best and worst teams is small. In the 2015 regular season, St. Louis Cardinals achieves the best winning percentage (0.617) while Philadelphia Phillies had the worse (0.389). The difference of 0.228 is much lower than for basketball ($0.890 - 0.122 = 0.768$ according to tables 4.2 and 4.3 for the 2016 season).

We believe that the ih -index provides a strong incentive system for players/teams since it potentially rewards more wins against high-ranked players/teams than against low-ranked players/teams. Bonus system exist or have existed in many

sports rankings, and some of them are also based on the strength of the opponents. Between 1994 and 1999, the ATP ranking was including bonuses depending on the current ranking of the defeated players. For instance, a win against world number one was associated with a 50 points bonus (doubled at a grand slam event), which was a substantial amount. Another example is the Elo rating system, used in chess but also for calculating the FIFA Women's World Rankings, which incorporates bonuses according to the difference in ranking between two opponents.

4.5 Conclusion

The main purpose of this article was to introduce a new kind of index for measuring the productivity of scholars, by allowing multi-dimensions. Even though our study has been focused on the h -index, we think that extensions of other indices in the same vein would deserve interest. Another task which we leave for future works is to find other applications for these multidimensional indices. Sports ranking have provided an interesting example in this article. To the best of our knowledge, [Hovden \(2013\)](#) is the only other related work based on the h -index, which is used to evaluate the performance of video channels on YouTube.

Bibliography

- Bornmann, L., Mutz, R., Hug, S. E., Daniel, H.-D., 2011. A multilevel meta-analysis of studies reporting correlations between the h index and 37 different h index variants. *Journal of Informetrics* 5, 346–349.
- Bouyssou, D., Marchand, T., 2014. An axiomatic approach to bibliometric rankings and indices. *Journal of Informetrics* 8, 449–477.
- Bouyssou, D., Marchand, T., 2016. Ranking authors using fractional counting of citations: An axiomatic approach. *Journal of Informetrics* 10, 183–199.
- Bring, J., Thuresson, M., 2011. Three points for a win in soccer: Is it fair? *CHANCE* 24, 47–53.
- Chambers, C. P., Miller, A. D., 2014. Scholarly influence. *Journal of Economic Theory* 151, 571–583.
- Clarke, S. R., Dyte, D., 2000. Using official ratings to simulate major tennis tournaments. *International Transactions in Operational Research* 7, 585–594.
- Dahl, G., 2012. A matrix-based ranking method with application to tennis. *Linear Algebra and its Applications* 437, 26–36.
- del Corrala, J., Prieto-Rodríguez, J., 2010. Are differences in ranks good predictors for Grand Slam tennis matches? *International Journal of Forecasting* 26, 551–563.
- Dilger, A., Geyer, H., 2009. Are three points for a win really better than two? a comparison of german soccer league and cup games. *Journal of Sports Economics* 10, 305–318.
- García-Pérez, M. A., 2009. A multidimensional extension to Hirsch’s h -index. *Scientometrics* 81, 779–785.

- García-Pérez, M. A., 2012. An extension of the h index that covers the tail and the top of the citation curve and allows ranking researchers with similar h. *Journal of Informetrics* 6, 689–699.
- Guedes, J. C., Machado, F. S., 2002. Changing rewards in contests: Has the three-point rule brought more offense to soccer? *Empirical Economics* 27, 607–630.
- Hirsch, J. E., 2005. An index to quantify an individual's scientific research output. *Proceedings of the National Academy of Sciences* 102, 16569–16572.
- Hovden, R., 2013. Bibliometrics for internet media: Applying the h-index to YouTube. *Journal of the Association for the Information Science and Technology* 64, 2326–2331.
- Hwang, Y.-A., 2013. An axiomatization of the Hirsch-index without adopting monotonicity. *Applied Mathematics & Information Sciences. An International Journal* 7, 1317–1322.
- Kongo, T., 2014. An alternative axiomatization of the Hirsch index. *Journal of Informetrics* 8, 252–258.
- Lenten, L. J. A., 2016. Mitigation of perverse incentives in professional sports leagues with reverse-order drafts. *Review of Industrial Organization* 49, 25–41.
- Miroiu, A., 2013. Axiomatizing the Hirsch index: Quantity and quality disjoined. *Journal of Informetrics* 7, 10–15.
- Motomura, A., Roberts, K. V., Leeds, D. M., Leeds, M. A., 2016. Does it pay to build through the draft in the national basketball association? *Journal of Sports Economics* 17, 501–516.
- Palacios-Huerta, I., Volij, O., 2004. The measurement of intellectual influence. *Econometrica* 72, 963–977.
- Palacios-Huerta, I., Volij, O., 2014. Axiomatic measures of intellectual influence. *International Journal of Industrial Organization* 34, 85–90.
- Perry, M., Reny, P. J., 2016. How to count citations if you must, forthcoming in *American Economic Review*.

- Price, J., Soebbing, B. P., Berri, D., Humphreys, B. R., 2010. Tournament incentives, league policy, and NBA team performance revisited. *Journal of Sports Economics* 11, 117–135.
- Quesada, A., 2010. More axiomatics for the Hirsch index. *Scientometrics* 82, 413–418.
- Quesada, A., 2011a. Axiomatics for the Hirsch index and the Egghe index. *Journal of Informetrics* 5, 476–480.
- Quesada, A., 2011b. Further characterizations of the Hirsch index. *Scientometrics* 87, 107–114.
- Ruiz, J. L., Pastor, D., Pastor, J. T., 2013. Assessing professional tennis players using data envelopment analysis (DEA). *Journal of Sports Economics* 14, 276–302.
- Sen, A., 1973. *On Economic Inequality*. Oxford University Press.
- Taylor, B. A., Trogdon, J. G., 2002. Tournament incentives in the national basketball association. *Journal of Labour Economics* 20, 23–41.
- Woeginger, G. J., 2008a. An axiomatic characterization of the Hirsch-index. *Mathematical Social Sciences* 56, 224–232.
- Woeginger, G. J., 2008b. A symmetry axiom for scientific impact indices. *Journal of Informetrics* 2, 298–303.

General conclusion and future work

This thesis mainly deals with the axiomatic approach and its use in cooperative game theory and scientometrics. This approach claims to root solutions to some desired properties, called axioms. Through the four chapters presented here, many different axioms have been defined and combined in order to characterize evaluative functions (allocation rules or indices).

During my years of thesis, some of my works has proved not directly publishable and getting back to these unfinished drafts will be a first objective. Here I rather list briefly some new ideas for future work: one challenging objective will be to use the axiomatic approach in the area of grading: how should a fair marking scheme be designed according to the exercises actually tackled by the students in an examination? This question haunts me regularly in the exhausting moments of marking copies. Another work, already in progress, concerns the question of defining fair allocations of rewards in a multi-level marketing and trying to apply this literature to scientometrics: the citation process for scholars can be seen as a long chain of referrals. Thus ancestors of a research branch should be rewarded in some way, even if their work are not directly quoted: how to create an index according to this tree-like model?

Appendix: Declarations of authorship

I hereby declare that I have written this thesis without any help from others and without the use of documents and aids other than those stated clearly in this document. Furthermore, I have mentioned all used sources and have cited them correctly according to the citation rules defined by the Chair of Economics and Information Systems. Moreover, I confirm that the paper at hand was not submitted in this or similar form at another examination office, nor has it been published before. With my signature I explicitly approve that HHL will use an internet-based plagiarism detector which screens electronic text files and looks for similar pieces on open-access websites as well as similarities in work previously submitted.

Besançon,
September 1st, 2016

A handwritten signature in dark ink, appearing to read 'S. Ferrières', with a long horizontal stroke extending to the right.

Sylvain Ferrières

Declaration of the co-authors concerning Mr. Ferrières's contributions

We hereby declare that the doctoral student Ferrières, Sylvain has completely fulfilled his parts in the co-authored studies “Axiomatic characterizations under players nullification” (chapter 1), “The proportional Shapley value and an application” (chapter 3), and “An axiomatization of the iterated h -index and applications to sport rankings” (chapter 4) as follows. Moreover, Mr. Ferrières is sole author of study “Nullified equal loss property and equal division values” (chapter 2).

Chapter	Name	Contributions
1	Sylvain Ferrières (joint work with Sylvain Béal, Eric Rémila and Philippe Solal)	Main author, discovery of the general idea of the article, production of results, writing of proofs, writing of the article.
2	Sylvain Ferrières	Author's independent research, sole author.
3	Sylvain Ferrières (joint work with Sylvain Béal, Eric Rémila and Philippe Solal)	Main author, discovery of the general idea of the article, economic application of the results, production of results, writing of proofs, proof reading.
4	Sylvain Ferrières (joint work with Sylvain Béal, Eric Rémila and Philippe Solal)	Main author, discovery of the general idea of the article, data collection and management, computational implementation, writing of the article.

The parts Mr. Ferrières has independently contributed were essential for the completion of these studies.

Besançon,
November 6, 2016



Sylvain Béal

Saint-Etienne,
November 9, 2016



Eric Rémila

Saint-Etienne,
November 9, 2016



Philippe Solal

General bibliography

- Aadland, D., Kolpin, V., 1998. Shared irrigation costs: an empirical and axiomatic analysis. *Mathematical Social Sciences* 35 (2), 203–218.
- Alain, É. C., 1928. *Propos sur le bonheur*. Gallimard, Paris.
- Ambec, S., Ehlers, L., 2008. Sharing a river among satiable agents. *Games and Economic Behavior* 64 (1), 35–50.
- Ambec, S., Sprumont, Y., 2002. Sharing a river. *Journal of Economic Theory* 107 (2), 453–462.
- Arrow, K., 1963. *Social choice and individual values*. Wiley, New York.
- Aumann, R. J., 1959. Acceptable points in general cooperative n -person games. *Contributions to the Theory of Games* 4.
- Aumann, R. J., 1960. Linearity of unrestrictedly transferable utilities. *Naval Research Logistics Quarterly* 7 (3), 281–284.
- Aumann, R. J., 1974. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics* 1 (1), 67–96.
- Aumann, R. J., 1989. Game theory. In: *Game Theory*. Springer, pp. 1–53.
- Aumann, R. J., Serrano, R., 2008. An economic index of riskiness. *Journal of Political Economy* 116 (5), 810–836.
- Baker, M., Associates, 1965. Runway cost impact study. Report presented to the Association of Local Transport Airlines, Jackson, Mississippi.
- Balinski, M., Young, H. P., 1982. Fair representation in the european parliament. *Journal of Common Market Studies* 20 (4), 361–373.

- Barbera, S., Jackson, M., 1994. A characterization of strategy-proof social choice functions for economies with pure public goods. *Social Choice and Welfare* 11 (3), 241–252.
- Béal, S., Casajus, A., Hüttner, F., Rémila, E., Solal, P., 2014. Solidarity within a fixed community. *Economics Letters* 125 (3), 440–443.
- Béal, S., Deschamps, M., Solal, P., 2016. Comparable axiomatizations of two allocation rules for cooperative games with transferable utility and their subclass of data games. *Journal of Public Economic Theory* forthcoming.
- Béal, S., Ferrières, S., Rémila, E., Solal, P., 2016. Axiomatic characterizations under players nullification. *Mathematical Social Sciences* 80, 47–57.
- Bernheim, B. D., 1998. Rational strategic choice revisited. *The Scandinavian Journal of Economics* 100 (2), 537–541.
- Bloch, F., Jackson, M. O., Tebaldi, P., 2016. Centrality measures in networks. Available at SSRN 2749124.
- Bouyssou, D., Marchant, T., 2014. An axiomatic approach to bibliometric rankings and indices. *Journal of Informetrics* 8 (3), 449–477.
- Casajus, A., Huettner, F., 2013. Null players, solidarity, and the egalitarian Shapley values. *Journal of Mathematical Economics* 49 (1), 58–61.
- Cook, K., Hegtvedt, K., 1983. Distribution justice, equity and equality. *Annual Reviews of Sociology* 9, 217–241.
- Derks, J. J., Haller, H. H., 1999. Null players out? Linear values for games with variable supports. *International Game Theory Review* 1, 301–314.
- Deutsch, M., 1975. Equity, equality, and need: What determines which value will be used as the basis of distributive justice? *Journal of Social issues* 31 (3), 137–149.
- Dubey, P., 1975. On the uniqueness of the Shapley value. *International Journal of Game Theory* 4 (3), 131–139.
- Emek, Y., Karidi, R., Tennenholtz, M., Zohar, A., 2011. Mechanisms for multi-level marketing. *Proceedings of the 12th ACM conference on Electronic commerce*, ACM, 209–218.

- Foster, J. E., 2006. Poverty indices. In: Poverty, Inequality and Development. Springer, pp. 41–65.
- García-Pérez, M., 2009. A multidimensional extension to Hirsch's h -index. *Scientometrics* 81 (3), 779–785.
- Gefeller, O., Land, M., Eide, G. E., 1998. Averaging attributable fractions in the multifactorial situation: Assumptions and interpretation. *Journal of Clinical Epidemiology* 51 (5), 437–441.
- Gómez-Rúa, M., Vidal-Puga, J., 2010. The axiomatic approach to three values in games with coalition structure. *European Journal of Operational Research* 207 (2), 795–806.
- Grabisch, M., 1996. The representation of importance and interaction of features by fuzzy measures. *Pattern Recognition Letters* 17 (6), 567–575.
- Graham, D. A., Marshall, R. C., Richard, J.-F., 1990. Differential payments within a bidder coalition and the Shapley value. *The American Economic Review* 80 (3), 493–510.
- Gul, F., 1989. Bargaining foundations of Shapley value. *Econometrica* 57 (1), 81–95.
- Hardin, G., 1968. The Tragedy of the Commons. *Science* 162, 1243–1248.
- Harsanyi, J., 1959. A bargaining model for the cooperative n -person game. In: Tucker, A. W., Luce, R. D. (Eds.), *Contributions to the Theory of Games (AM-40)*. Vol. 4. Princeton University Press, pp. 325–355.
- Harsanyi, J. C., 1968. Games with incomplete information played by "bayesian" players part II. bayesian equilibrium points. *Management Science* 14 (5), 320–334.
- Hart, S., Mas-Colell, A., 1989. Potential, value, and consistency. *Econometrica* 57 (3), 589–614.
- Hart, S., Mas-Colell, A., 1996. Bargaining and value. *Econometrica* 64 (2), 357–380.
- Hirsch, J. E., 2005. An index to quantify an individual's scientific research output. *Proceedings of the National academy of Sciences of the United States of America*, 16569–16572.

- Holzman, R., Moulin, H., 2013. Impartial nominations for a prize. *Econometrica* 81 (1), 173–196.
- Israeli, O., 2007. A Shapley-based decomposition of the r-square of a linear regression. *The Journal of Economic Inequality* 5 (2), 199–212.
- Joosten, R. A. M. G., 1996. Dynamics, equilibria, and values. Ph.D. thesis, Maastricht university.
- Kamijo, Y., Kongo, T., 2010. Axiomatization of the Shapley value using the balanced cycle contributions property. *International Journal of Game Theory* 39 (4), 563–571.
- Kelly, J., 1978. Arrow impossibility theorems. Academic Press, New York.
- Kreps, D., Wilson, R., 1982. Sequential equilibria. *Econometrica* 50 (4), 863–894.
- Laruelle, A., Valenciano, F., 2008. Potential, value, and coalition formation. *TOP* 16 (1), 73–89.
- Littlechild, S. C., Owen, G., 1973. A simple expression for the Shapley value in a special case. *Management Science* 20 (3), 370–372.
- Lozano, S., Moreno, P., Adenso-Díaz, B., Algaba, E., 2013. Cooperative game theory approach to allocating benefits of horizontal cooperation. *European Journal of Operational Research* 229 (2), 444–452.
- Maniquet, F., 2003. A characterization of the Shapley value in queueing problems. *Journal of Economic Theory* 109 (1), 90–103.
- Monderer, D., Samet, D., Shapley, L. S., 1992. Weighted values and the core. *International Journal of Game Theory* 21 (1), 27–39.
- Moretti, S., Patrone, F., 2008. Transversality of the Shapley value. *TOP* 16 (1), 1–41.
- Moretti, S., Patrone, F., Bonassi, S., 2007. The class of microarray games and the relevance index for genes. *TOP* 15 (2), 256–280.
- Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory* 16 (3), 161–186.

- Moulin, H., 1992. An application of the Shapley value to fair division with money. *Econometrica* 60 (6), 1331–1349.
- Moulin, H., Shenker, S., 1992. Serial cost sharing. *Econometrica* 60 (5), 1009–1037.
- Myerson, R. B., 1977a. Graphs and cooperation in games. *Mathematics of Operations Research* 2 (3), 225–229.
- Myerson, R. B., 1977b. Values of games in partition function form. *International Journal of Game Theory* 6 (1), 23–31.
- Myerson, R. B., 1980. Conference structures and fair allocation rules. *International Journal of Game Theory* 9 (3), 169–182.
- Myerson, R. B., 1991. *Game theory: analysis of conflict*. Harvard University.
- Naor, M., 2005. On fairness in the carpool problem. *Journal of Algorithms* 55 (1), 93–98.
- Nash, J. F., 1950. The bargaining problem. *Econometrica* 18, 155–162.
- Nash, J. F., 1951. Non-cooperative games. *Annals of Mathematics* 54, 286–295.
- Neyman, A., 1989. Uniqueness of the Shapley value. *Games and Economic Behavior* 1, 116–118.
- Owen, G., 1977. Values of games with a priori unions. In: *Mathematical economics and game theory*. Springer, pp. 76–88.
- Palacios-Huerta, I., Volij, O., 2004. The measurement of intellectual influence. *Econometrica* 72 (3), 963–977.
- Peleg, B., Tijs, S., 1996. The consistency principle for games in strategic form. *International Journal of Game Theory* 25 (1), 13–34.
- Pérez-Castrillo, D., Wettstein, D., 2001. Bidding for the surplus: a non-cooperative approach to the Shapley value. *Journal of Economic Theory* 100 (2), 274–294.
- Rackham, H., 1934. *Aristotle in 23 Volumes*. Vol. 19. Cambridge, MA: Harvard University Press.

- Rahwan, T., Naroditskiy, V., Michalak, T., Wooldridge, M., Jennings, N. R., 2014. Towards a fair allocation of rewards in multi-level marketing. arXiv preprint: 1404.0542.
- Ramamurthy, K., 1990. Coherent structures and simple games, Vol. 6 of Theory and Decision Library. Series C: Game Theory. Mathematical Programming and Operations Research. Kluwer Academic Publishers Group, Dordrecht.
- Rawls, J., 2009. A theory of justice, *revised edition*. Harvard university press.
- Roth, A. E., 1988. The Shapley value: essays in honor of Lloyd S. Shapley. Cambridge University Press.
- Selten, R., 1965. Spieltheoretische behandlung eines oligopolmodells mit nachfragerträgeit: Teil i: Bestimmung des dynamischen preisgleichgewichts. Zeitschrift für die gesamte Staatswissenschaft/Journal of Institutional and Theoretical Economics 121 (2), 301–324.
- Sen, A., 1976. Poverty: an ordinal approach to measurement. Econometrica 44 (2), 219–231.
- Shapley, L. S., 1953. Additive and non-additive set functions. Ph.D. thesis.
- Shapley, L. S., 1953. A value for n -person games. In: Contribution to the Theory of Games vol. II (H.W. Kuhn and A.W. Tucker eds). Annals of Mathematics Studies 28. Princeton University Press, Princeton.
- Shapley, L. S., Shubik, M., 1954. A method for evaluating the distribution of power in a committee system. American Political Science Review 48 (3), 787–792.
- Shapley, L. S., Shubik, M., 1967. Ownership and the production function. The Quarterly Journal of Economics, 88–111.
- Shubik, M., 1962. Incentives, decentralized control, the assignment of joint costs and internal pricing. Management science 8 (3), 325–343.
- Suijs, J., Borm, P., 1999. Stochastic cooperative games: superadditivity, convexity, and certainty equivalents. Games and Economic Behavior 27 (2), 331–345.
- Thomson, W., 2001. On the axiomatic method and its recent applications to game theory and resource allocation. Social Choice Welfare 18, 327–386.

- Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences* 45 (3), 249–297.
- Thomson, W., 2010. Fair allocation rules. In: *Handbook of Social Choice and Welfare* (K. Arrow, A. Sen, and K. Suzumura, eds). North-Holland, Amsterdam, New York, pp. 393–506.
- Thomson, W., 2011. Consistency and its converse: an introduction. *Review of Economic Design* 15 (4), 257–291.
- Thomson, W., 2012. On the axiomatics of resource allocation: Interpreting the consistency principle. *Economics and Philosophy* 28 (3), 385–421.
- Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. *Mathematical Social Sciences* 74, 41–59.
- Thomson, W., mimeo 2006; revised July 2013. Airport problems and cost allocation. Rochester Center for Economic Research, Working Paper.
- Thrall, R. M., Lucas, W. F., 1963. N-person games in partition function form. *Naval Research Logistics Quarterly* 10 (1), 281–298.
- Tricot, J., 1979. *Éthique à Nicomaque*. J. Vrin, Paris.
- van den Brink, R., 2007. Null or nullifying players: the difference between the Shapley value and equal division solutions. *Journal of Economic Theory* 136 (1), 767–775.
- van den Brink, R., Funaki, Y., 2009. Axiomatizations of a class of equal surplus sharing solutions for TU-games. *Theory and Decision* 67 (3), 303–340.
- Young, H. P., 1985. Monotonic solutions of cooperative games. *International Journal of Game Theory* 14 (2), 65–72.

Four essays on the axiomatic method: cooperative game theory and scientometrics

Ferrières Sylvain

Résumé : La thèse propose quatre contributions sur la méthode axiomatique. Les trois premiers chapitres utilisent le formalisme des jeux coopératifs à utilité transférable. Dans les deux premiers chapitres, une étude systématique de l'opération de nullification est menée. Les axiomes de retraits sont transformés en axiomes de nullification. Des caractérisations existantes de règles d'allocation sont revisitées, et des résultats totalement neufs sont présentés. Le troisième chapitre introduit et caractérise une valeur de Shapley proportionnelle, où les dividendes d'Harsanyi sont partagés en proportion des capacités des singletons concernés. Le quatrième chapitre propose une variante multi-dimensionnelle de l'indice de Hirsch. Une caractérisation axiomatique et une application aux classements sportifs sont fournies.

Mots-clés : Méthode axiomatique, Théorie des jeux coopératifs, Scientométrie, Nullification, Valeur de Shapley proportionnelle, Indice de Hirsch itéré.

Abstract: The dissertation provides four contributions on the axiomatic method. The first three chapters deal with cooperative games with transferable utility. In the first two chapters, a systematic study of the nullification operation is done. The removal axioms are translated into their nullified counterparts. Some existing characterizations are revisited, and completely new results are presented. The third chapter introduces and characterizes a proportional Shapley value in which the Harsanyi dividends are shared in proportion to the stand-alone worths of the concerned players. The fourth chapter proposes a multi-dimensional variant of the Hirsch index. An axiomatic characterization and an application to sports rankings are provided.

Keywords: Axiomatic method, Cooperative Game Theory, Scientometrics, Nullification, Proportional Shapley value, Iterated Hirsch-index.